1. If p divides q and p divides r, then p divides $q + r$.

Proof:

If p divides q then, $q = p \cdot n$ for some $n \in \mathbb{Z}$.

If p divides r then $r = p \cdot m$ for some $m \in \mathbb{Z}$.

Then, to show that p divides $q+r$.

$$q + r = p \cdot n + p \cdot m$$

$$q + r = p(n+m)$$

Since, $(n+m)$ is an integer too by closure of integer by addition, we can say that

p divides $q+r$ by divisibility. $\square$

Excellent

2. $P \Rightarrow Q$ is logically equivalent to its contrapositive.

The contrapositive of $P \Rightarrow Q$ is $\neg Q \Rightarrow \neg P$

P	Q	$P \overset{*}{\Rightarrow} Q$	$\neg Q$	$\neg P$	$\neg Q \overset{*}{\Rightarrow} \neg P$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

The truth values for $*$ are same under possible circumstances.
therefore $P \Rightarrow Q$ is logically equivalent to its contrapositive. $\square$

Excellent

3. If $a \equiv_5 4$, then $a^2 \equiv_5 1$.

If $a \equiv_5 4$, then $5 \mid (4-a)$, so $(4-a) = 5 \cdot n$ for some $n \in \mathbb{Z}$.
Rearranging, we see that $a = -5n + 4$. we wanna know something about a^2, so let's square!

$a = -5n + 4$

$a^2 = (-5n+4)^2$

$a^2 = 25n^2 - 40n + 16$

$a^2 = 5(5n^2 - 8n + 3) + 1$

Since $(5n^2 - 8n + 3) \in \mathbb{Z}$ by closure of integers under multiplication and addition, let's call it the integer c.

So $a^2 = 5c + 1$.

Rearranging, $1 - a^2 = -5c$, which tells us that $5 \mid (1-a^2)$, and therefore $a^2 \equiv_5 1$. $\square$

Nice!

4. $\sqrt{3}$ is irrational.

well, Suppose $\sqrt{3}$ was a rational number such that we could write it as $\frac{p}{q}$ with $q \neq 0$ for some $p, q \in \mathbb{Z}$, where p/q has been reduced so that p and q share no common factors.

$\sqrt{3} = \frac{p}{q} \iff 3 = \frac{p^2}{q^2} \iff p^2 = 3q^2$.

$p^2 = 3q^2$ tells us that p^2 is threven, and based on previous proofs, that means that p is also threven. Now let's call p the threven number $p = 3r$ for some $r \in \mathbb{Z}$.

$p^2 = 3q^2 \iff (3r)^2 = 3q^2 \iff 9r^2 = 3q^2 \iff 3r^2 = q^2$.

$3r^2 = q^2$ tells us that q^2 is threven, and based on previous proofs, q is also threven. But if p and q are both threven, then they share a common factor of 3, contradicting our supposition that $\frac{p}{q}$ had been reduced to the point of no common factors. Therefore, $\sqrt{3}$ is irrational by contradiction. $\square$

Great

$1+3+5=9$ ✓ $3^2=9$ ✓

5. For any $n \in \mathbb{Z}^+$,

$$\sum_{i=1}^{n}(2i-1) = n^2$$

Let's proceed with induction:

Base Case: Test $n=1$. $\sum_{i=1}^{1}(2i-1) = 1 = 1^2 = 1$ True ✓

Induction Hypothesis: Assume for some $k \in \mathbb{Z}^+$, $\sum_{i=1}^{k}(2i-1) = k^2$

Work: Want to show $\sum_{i=1}^{k+1}(2i-1) = (k+1)^2$

$\sum_{i=1}^{k+1}(2i-1) = \sum_{i=1}^{k}(2i-1) + (2(k+1)-1)$

By our I.H, $\sum_{i=1}^{k}(2i-1) = k^2$

So: $k^2 + 2(k+1) - 1 = k^2 + 2k + 2 - 1 = k^2 + 2k + 1 =$
$(k+1)(k+1) = (k+1)^2$

Great!

$\therefore$ Since our base case of $n=1$ is true, and by our induction hypothesis, $\sum_{i=1}^{k+1}(2i-1) = (k+1)^2$, by mathematical induction, for any $n \in \mathbb{Z}^+$, $\sum_{i=1}^{n}(2i-1) = n^2$ holds true