Examlet 2a **Foundations of Advanced Math** **2/23/24**

1. Write each of the following in as simple a way as possible:

   (a) $(5, 10) - [6, 12]$

   $x \in (5, 10)$
   $x \notin [6, 12]$ $= \boxed{x \in (5, 6)}$

   (b) $[5, 10] - [6, 12]$

   $x \in [5, 10]$
   $x \notin [6, 12]$ $= \boxed{x \in [5, 6)}$

   Good

   (c) $(5, 10) \cap [6, 12]$

   $x \in (5, 10) \cap$
   $x \in [6, 12]$ $= \boxed{x \in [6, 10)}$

   Circle T or F for each of the following statements:

   (d) $\emptyset \subseteq \{0, 1, 2\}$ T F

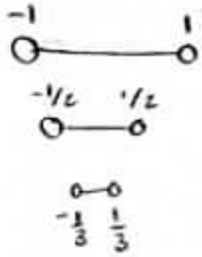
   (e) $\{\emptyset\} \subseteq \{0, 1, 2\}$ T F in all

2. (a) What is $\bigcap_{n\in\{1,2,3\}} \left(-\frac{1}{n}, \frac{1}{n}\right)$?

$n=1 \quad \left(-\frac{1}{1}, \frac{1}{1}\right) \Rightarrow (-1, 1)$

$n=2 \quad \left(-\frac{1}{2}, \frac{1}{2}\right)$

$n=3 \quad \left(-\frac{1}{3}, \frac{1}{3}\right)$



$\bigcap_{n\in\{1,2,3\}} \left(-\frac{1}{n}, \frac{1}{n}\right) = \underline{\left(-\frac{1}{3}, \frac{1}{3}\right)}$

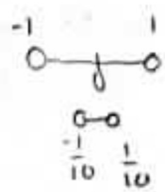
(b) What is $\bigcup_{n\in\{1,2,3\}} \left(-\frac{1}{n}, \frac{1}{n}\right)$? in some

$\underline{(-1, 1)}$

(c) What is $\bigcap_{n\in\mathbf{Z}^+} \left(-\frac{1}{n}, \frac{1}{n}\right)$? $= \underline{\{0\}}$

$n=1 \quad (-1, 1)$

$n=10 \quad \left(-\frac{1}{10}, \frac{1}{10}\right)$



Great

(d) What is $\bigcup_{n\in\mathbf{Z}^+} \left(-\frac{1}{n}, \frac{1}{n}\right)$? $= \underline{(-1, 1)}$

$n=1000 \quad \left(-\frac{1}{1000}, \frac{1}{1000}\right)$

$n=1 \quad (-1, 1)$

$n=0 \quad (0, 0)$

3. $\left(\bigcup_{i\in I} B_i\right)' = \bigcap_{i\in I} B_i'$.

Well, take an element of the set on the left.

$$x \in \left(\bigcup_{i\in I} B_i\right)' \iff \neg\, x \in \left(\bigcup_{i\in I} B_i\right)$$

$$\iff \neg\left(\exists\, i \in I \text{ for which } x \in B_i\right)$$

By the box on p.12

$$\iff \forall i \in I,\ \neg\, x \in B_i$$

$$\iff \forall i \in I,\ x \in B_i'$$

$$\iff x \in \bigcap_{i\in I} B_i'$$

This shows $\left(\bigcup_{i\in I} B_i\right)' \subseteq \bigcap_{i\in I} B_i'$. Since each step is reversible we also have the reverse inclusion, so the sets are equal. $\square$

$c = \frac{1}{a}$ $c \le \frac{1}{a}$

$c > 0$ $\frac{1}{a} < 0$

$c = \frac{1}{a}$ $a > 0$

$ca = 1$ $\frac{1}{a} < 0$

$\frac{a}{a} < 0$

$1 < 0$

-3 1

4. (a) $\forall a, b, c, d \in \mathbb{R}, a < b$ and $c < d \Rightarrow \frac{a}{d} < \frac{b}{c}$

-1 -2

Counterexample:

$a = -3 \quad b = 1 \quad c = -2 \quad d = -1$

$a < b$: $-3 < 1$ True

$c < d$: $-2 < -1$ True

$\frac{a}{d} < \frac{b}{c}$: $\frac{-3}{-1} < \frac{1}{-2}$

$3 < -\frac{1}{2}$ False. □

Good

(b) $\forall a, b, c, d \in \mathbb{R}^+, a < b$ and $c < d \Rightarrow \frac{a}{d} < \frac{b}{c}$

Since $\forall a, b, c, d \in \mathbb{R}^+$, $0 < a$, $0 < b$, $0 < c$, $0 < d$. Therefore, by CMP, $a < b \rightarrow ac < bc$ and $c < d \rightarrow bc < bd$. Then, by Transitive Property, $ac < bd$.

Then, prove for $r > 0$, $\frac{1}{r} > 0$. - Suppose $r > 0$ but $\frac{1}{r} \le 0$. Then, by CMP, multiply both sides of $\frac{1}{r} \le 0$ by r to get $\frac{r}{r} \le 0$, which is $1 \le 0$. This contradicts our supposition, so if $r > 0$, $\frac{1}{r} > 0$. Therefore, since $c > 0$ and $d > 0$, $\frac{1}{c} > 0$ and $\frac{1}{d} > 0$.

By CMP: $ac < bd \rightarrow \frac{ac}{d} < b \rightarrow \frac{a}{d} < \frac{b}{c}$

(Multiply by $\frac{1}{d}$) (Multiply by $\frac{1}{c}$). □

Nice

5. $\forall x \in \mathbb{R}, -|x| \le x \le |x|$. (Lemma # 1)

$\rightarrow$ The definition of absolute value of x states that:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Now,

Case - I

$x \geq 0$

So, by definition, $|x| = x$ and we can say that $x \le |x|$.

$x \geq 0$

Now, adding $-x$ on both sides of $x \geq 0$,

$x + (-x) \geq 0 + (-x)$

or, $0 \geq -x$

or, $-x \le 0$

$\therefore -|x| = -x \le 0 \le x \le |x|$

Case - II

$x < 0$

So, by definition, $x < 0$ and we can say that $-x \le |x|$

$x < 0$

Now, adding $-x$ on both sides of $x < 0$,

$x + (-x) < 0 + (-x)$

$0 < -x$

$\therefore -|x| < x < 0 < -x \le |x|$

Hence, in both the cases $-|x| \le x \le |x|$ as desired.

Great!