1. Let $f, g : \mathbb{R} \to \mathbb{R}$, and suppose that f and g are both odd functions. Then $f \cdot g$ is an (even / odd – pick one and defend your claim) function.

Since $f,g:\mathbb{R}\to\mathbb{R}$ are both odd functions, $\forall x\in\mathbb{R}$, $f(x)=-f(-x)$ and $g(x)=-g(-x)$. So $h(x)=(f\cdot g)(x)=f(x)\cdot g(x)=(-f(-x))\cdot(-g(-x))$ $=f(-x)\cdot g(-x)=(f\cdot g)(-x)=h(-x)$. Since $h(x)=h(-x)$, the product of two odd functions is even by definition. □

Excellent!

2. If $f : A \to B$ and $g : B \to C$ are injective functions, then $g \circ f$ is injective.

Since f and g are both injective functions, it is known that $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$ and $g(b_1) = g(b_2) \Rightarrow b_1 = b_2$

Now, consider $g \circ f(a_1) = g \circ f(a_2)$

$= g(f(a_1)) = g(f(a_2))$ by def of composition

$\Rightarrow f(a_1) = f(a_2)$ since g is injective

$\Rightarrow a_1 = a_2$ since f is injective

$g \circ f(a_1) = g \circ f(a_2) \Rightarrow a_1 = a_2$

$g \circ f$ is injective when g and f are both injective

Great!

3. If $f, g : \mathbb{R} \to \mathbb{R}$ are both bounded, then:

(a) $f + g$ is bounded

By definition of bounded functions, we know that $\forall x \in R$, $\exists M, N \in R$ such that $|f(x)| \le M$ and $|g(x)| \le N$.

Then add these inequalities to get:

$$|f(x)| + |g(x)| \le M + N$$

Use the Triangle Inequality to get

$$|f(x) + g(x)| \le |f(x)| + |g(x)|.$$

Therefore, by the transitive property,

$$|f(x) + g(x)| \le M + N$$

$|f(x) + g(x)| = |f + g(x)|$ and $M + N \in R$ by Closure of Reals under Addition

Therefore, $|f + g(x)| \le M + N$ for $M + N \in R$ and $\forall x \in R$.

Therefore, $f + g$ is bounded by definition. □

(b) f/g is bounded

Counterexample:

$f(x) = \sin(x)$

$g(x) = \cos(x)$

Great!

$\forall x \in R$, $|f(x)| \le 1$ and $|g(x)| \le 1$, $1 \in R$.

However,

$$\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \frac{\sin(x)}{\cos(x)} = \tan(x)$$

$\tan(x) \not\le M$, $M \in R$, so $\frac{f}{g}(x) \not\le M$.

Therefore, $\frac{f}{g}$ is not a bounded function. □

$$f: \begin{cases} \frac{x+3}{3} & x \text{ is threven} \\ \frac{x+2}{3} & x \text{ is throdd} \\ \frac{x+1}{3} & x \text{ is throddodd} \end{cases}$$

0+3 3+3 6+3 | 1 4 7 | Threven
1 2 3 | 0 1 2 | $\frac{x+1}{3}$
$\varnothing$ 4 7 | 1 2 3

4. If A, B and C are denumerable sets with each pair disjoint, then $A \cup B \cup C$ is denumerable.

By definition of disjoint, no elements in A, B, or C can be found in either of the other two sets.

Then, for $a_n \in A$, $b_n \in B$, and $c_n \in C$, arrange the union $A \cup B \cup C$ (containing all elements in each set) in an alternating fashion. No elements will be repeated since the sets are disjoint, so the pattern below will not be broken.

$A \cup B \cup C : \{a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, c_3, \ldots\}$

Then each element in $A \cup B \cup C$ can be counted in order and assigned a unique, corresponding element in $\mathbb{N}$ according to the bijection $f: \mathbb{N} \to A \cup B \cup C$ below.

$$f: \begin{cases} a_{\frac{x+3}{3}} & \text{for } x \in \mathbb{N} \text{ is threven} \\ b_{\frac{x+2}{3}} & \text{for } x \in \mathbb{N} \text{ is throdd} \\ c_{\frac{x+1}{3}} & \text{for } x \in \mathbb{N} \text{ is throddodd} \end{cases}$$

Since there exists a bijection $f: \mathbb{N} \to A \cup B \cup C$, $\mathbb{N}$ is equipollent to $A \cup B \cup C$ and $A \cup B \cup C$ is equipollent to $\mathbb{N}$, making $A \cup B \cup C$ denumerable.

Great

5. (a) Any two countable sets are equipollent.

The proposition is false. Let $A = \{1, 2\}$ and $B = \{4, 5, 6\}$. A and B are both countable since they are subsets of $\mathbb{N}$. However, a bijection can not exist from A to B, since a function will not be surjective. If $f: A \to B$, and $f(1) = 4$ and $f(2) = 5$, or $f(2) = 6$ and $f(1) = 5$, or $f(2) = 4$ and $f(1) = 6$, then not all elements are mapped in B. So, the proposition is false by counterexample

A (b) Any two denumerable sets are equipollent.

Say our two sets are A and B. So, since they are denumerable $\exists f: A \to \mathbb{N}$ that is bijective, and $\exists g: B \to \mathbb{N}$ that is bijective. So, we know since g is bijective, it must have an inverse h which is also bijective, where $h: \mathbb{N} \to B$. Now, consider $h \circ f(a)$. Well, we know the composition of two bijective functions must be a bijection from previous exercise, so that means a bijection exists between the two sets, since $h \circ f(a)$ would be from A to B

$\therefore$ Any two denumerable sets are equipollent

Excellent!