Exam 3b   **Foundations of Advanced Math**   3/22/24

1. Let $f, g : \mathbb{R} \to \mathbb{R}$, and suppose that f and g are both odd functions. Then $f + g$ is an (even / odd – pick one and defend your claim) function.

if $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are both odd functions then by definition, $-f(x) = f(-x), x \in \mathbb{R}$ and $-g(x) = g(-x), x \in \mathbb{R}$.
Considering this, lets add f and g creating $(f+g)(x) = f(x) + g(x)$.
if we let $x = -x$ then $(f+g)(-x) = f(-x) + g(-x)$. Based on our definitions of odd functions, $f(-x) + g(-x) = -f(x) - g(x)$
$\Rightarrow f(-x) + g(-x) = -(f(x) + g(x)) \Rightarrow (f+g)(-x) = -(f+g)(x)$ which is odd by definition.

Great

2. If $f : A \to B$ and $g : B \to C$ are injective functions, then $g \circ f$ is injective.

We know that $f(a_1) = f(a_2) \to a_1 = a_2$
and $g(b_1) = g(b_2) \to b_1 = b_2$.
by definition of injectivity.
Then $g \circ f(x) = g(f(x))$.
Since g is injective, $g(f(a_1)) = g(f(a_2)) \to f(a_1) = f(a_2)$.
Since f is injective, $f(a_1) = f(a_2) \to a_1 = a_2$.
By the transitive property,
$g(f(a_1)) = g(f(a_2)) \to a_1 = a_2$,
so $g \circ f$ is injective by definition. $\square$

Good

3. If $f, g : \mathbb{R} \to \mathbb{R}$ are both bounded, then:

(a) $f + g$ is bounded

By definition of bounded functions, we know that $\forall x \in R$, $\exists M, N \in R$ such that $|f(x)| \leq M$ and $|g(x)| \leq N$.

Then add these inequalities to get:

$$|f(x)| + |g(x)| \leq M + N$$

Use the Triangle Inequality to get

$$|f(x) + g(x)| \leq |f(x)| + |g(x)|.$$

Therefore, by the transitive property,

$$|f(x) + g(x)| \leq M + N$$

$|f(x) + g(x)| = |f + g(x)|$ and $M + N \in R$ by Closure of Reals under Addition

Therefore, $|f + g(x)| \leq M + N$ for $M + N \in R$ and $\forall x \in R$.

Therefore, $f + g$ is bounded by definition. □

(b) f/g is bounded

Counterexample:

$f(x) = \sin(x)$

$g(x) = \cos(x)$

Great!

$\forall x \in R$, $|f(x)| \leq 1$ and $|g(x)| \leq 1$, $1 \in R$.

However,

$$\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \frac{\sin(x)}{\cos(x)} = \tan(x)$$

$\tan(x) \nleq M$, $M \in R$, so $\frac{f}{g}(x) \nleq M$.

Therefore, $\frac{f}{g}$ is not a bounded function. □

$$f : \begin{cases} \frac{x+3}{3} & x \text{ is threven} \\ \frac{x+2}{3} & x \text{ is throdd} \\ \frac{x+1}{3} & x \text{ is throddodd} \end{cases}$$

0+3 3+3 6+3 1 4 7 threven

0 1 2 $\frac{x+1}{3}$

1 2 3 1 2 3

ϕ 4 7

4. If A, B and C are denumerable sets with each pair disjoint, then $A \cup B \cup C$ is denumerable.

By definition of disjoint, no elements in A, B, or C can be found in either of the other two sets.

Then, for $a_n \in A$, $b_n \in B$, and $c_n \in C$, arrange the union $A \cup B \cup C$ (containing all elements in each set) in an alternating fashion. No elements will be repeated since the sets are disjoint, so the pattern below will not be broken.

$A \cup B \cup C : \{a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, c_3, \ldots\}$

Then each element in $A \cup B \cup C$ can be counted in order and assigned a unique, corresponding element in $\mathbb{N}$ according to the bijection $f : \mathbb{N} \to A \cup B \cup C$ below.

$$f : \begin{cases} a_{\frac{x+3}{3}} & \text{for } x \in \mathbb{N} \text{ is threven} \\ b_{\frac{x+2}{3}} & \text{for } x \in \mathbb{N} \text{ is throdd} \\ c_{\frac{x+1}{3}} & \text{for } x \in \mathbb{N} \text{ is throddodd} \end{cases}$$

Since there exists a bijection $f : \mathbb{N} \to A \cup B \cup C$, $\mathbb{N}$ is equipollent to $A \cup B \cup C$ and $A \cup B \cup C$ is equipollent to $\mathbb{N}$, making $A \cup B \cup C$ denumerable.

Great

(Definition of equipollent - $\exists$ a bijection)

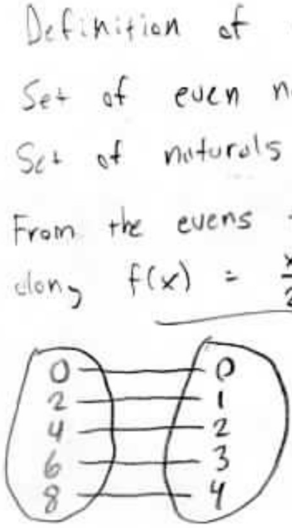
5. (a) The set of even natural numbers is denumerable.

Definition of denumerable: Equipollent to $\mathbb{N}$.

Set of even naturals: $\{0, 2, 4, 6, 8 \ldots \infty\}$

Set of naturals: $\{0, 1, 2, 3, 4 \ldots\}$

From the evens to the naturals, there is a bijection along $f(x) = \frac{x}{2}$. $\frac{0}{2} = 0$, $\frac{2}{2} = 1$, $\frac{4}{2} = 2$, $\frac{6}{2} = 3$, etc.



$f(x) = \frac{x}{2}$

Thus, there is a bijection between the evens and $\mathbb{N}$, meaning the even natural numbers are equipollent to $\mathbb{N}$, which by definition means the set of even natural numbers is denumerable.

(b) The set of irrational numbers is uncountable.

We've established previously the union of two countable sets is countable. We also know the set $\mathbb{R}$ is uncountable via Cantor's Theorem. (We also know $\mathbb{Q}$ is countable).

Let's say the set of irrational numbers is countable.

We know $\mathbb{R}$ = the set of rational numbers $\mathbb{Q}$ $\cup$ the set of irrational numbers.

If the irrationals were countable, $\mathbb{R}$ would be the union of two countable sets, meaning $\mathbb{R}$ is countable.

However, we've established $\mathbb{R}$ is already uncountable. Nice!

So we have a contradiction.

Thus, since we know $\mathbb{Q}$ is countable, that means the irrationals must be uncountable. TRUE. $\square$