Exam 1 Key

Calculus IV Exam 1 Spring 1999 2/4/99

Each problem is worth 10 points. Show all work for full credit. Please circle all answers and keep your work as legible as possible. No animals were harmed in the making of this exam.

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1. Show that $\lim_{(x,y)\to(0,0)} \frac{3xy}{5x^2 + 7y^2}$ does not exist.

the line $x=0$: $\lim_{(0,y)\to(0,0)} \frac{3(0)y}{0+7y^2} = \lim_{(0,y)\to(0,0)} \frac{0}{7y^2} = 0$

the line $y=x$: $\lim_{(x,x)\to(0,0)} \frac{3xx}{5x^2+7x^2} = \lim_{(x,x)\to(0,0)} \frac{3\cancel{x}^2}{12\cancel{x}^2} = \lim_{(x,x)\to(0,0)} \frac{3}{12} = \frac{3}{12} = \frac{1}{4}$

Since the limits do not equal each other, the limit does not exist.

Great

2. For reasons beyond human comprehension, the radius of a big right circular cone is increasing at a rate of 1.8 in/s while its height is decreasing at a rate of 2.5 in/s. At what rate is the volume of the cone changing when the radius is 120 in. and the height is 140 in.? [Hint: $V_{cone} = \dfrac{\pi r^2 h}{3}$]

W

$\frac{dr}{dt} = 1.8 \quad \frac{dh}{dt} = -2.5 \quad r = 120 \quad h = 140$

$V_{cone} = \frac{\pi r^2 h}{3}$ ⇒ Let us impliment the CHAIN RULE ⇒

$$\begin{array}{ccc} & V & \\ \frac{\partial V}{\partial r} \swarrow & & \searrow \frac{\partial V}{\partial h} \\ r & & h \\ \frac{dr}{dt} \downarrow & & \downarrow \frac{dh}{dt} \\ t & & t \end{array}$$

* [redacted]

Find $\frac{\partial V}{\partial r}$: $\frac{\partial V}{\partial r} = \left(\frac{\pi r^2}{3}\right) \cdot (0)$ since $h \to 0$ because treated as constant $+ h\left(\frac{2\pi}{3} r\right)$

Find $\frac{\partial V}{\partial h}$: $\frac{\partial V}{\partial h} = \left(\frac{\pi r^2}{3}\right) + h \cdot (0)$ same with r

Good

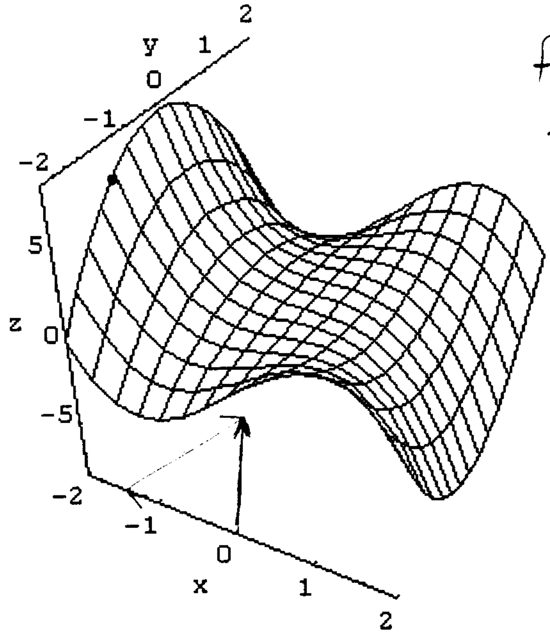
⇒ Now we put the partials back into the $*\frac{dV}{dt}$ equation and plug in values of $h, r, \frac{dr}{dt}, \frac{dh}{dt}$.

$$\Rightarrow \frac{dV}{dt} = \frac{2\pi r h}{3} \cdot \frac{dr}{dt} + \frac{\pi r^2}{3} \cdot \frac{dh}{dt} = \frac{2\pi (120)(140)}{3} \cdot (1.8) + \frac{\pi (120)^2}{3} \cdot (-2.5) = 63334.51 - 37699.1$$

≈ 25635.39 in^3 [redacted]

C 3. Compute the directional derivative of the function f(x,y) = xy² - x³ in the direction of the vector <-3,4> from the point (-1,-2).

$\vec{u} = \frac{\langle -3,4 \rangle}{5} = \langle -\frac{3}{5}, \frac{4}{5} \rangle$



W

$f_x(x,y) = y^2 - 3x^2$ Plug $(-1,-2)$ $= -1$

$f_y(x,y) = 2xy$ $\quad 4$

$D_u = \langle f_x(x,y), f_y(x,y) \rangle \cdot \vec{u}$

$= \langle -1, 4 \rangle \cdot \langle -\frac{3}{5}, \frac{4}{5} \rangle$

$= -\frac{3}{5} + \frac{16}{5}$

$= \boxed{\frac{13}{5}}$

4. Find the maximum rate of change of the function $f(x,y) = xy^2 - x^3$ at the point (1, 2) and the direction in which that maximum change occurs.

10

First get gradient

$$f_x = y^2 - 3x^2$$

$$f_y = 2xy \quad \nabla f(x,y) = \langle y^2 - 3x^2, 2xy \rangle$$

Plug in the pt.

$$\nabla f(1,2) = \langle 2^2 - 3(1)^2, 2(1)(2) \rangle$$

$$= \langle 1, 4 \rangle$$

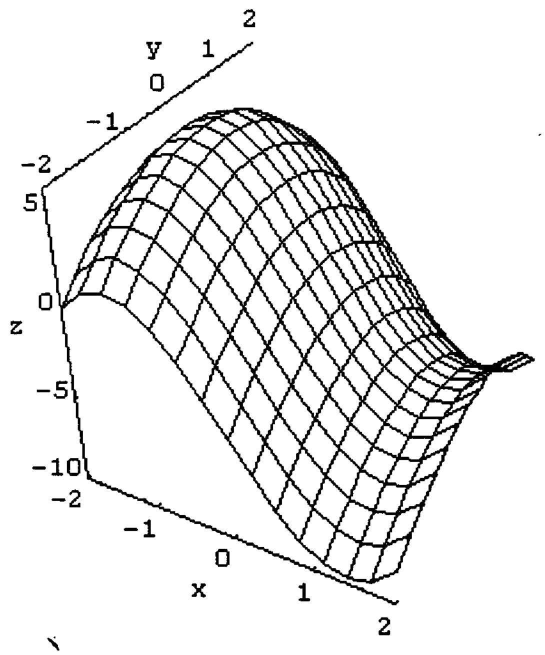
Next get the length of the gradient

$$\sqrt{1^2 + 4^2} = \sqrt{17}$$

Maximum Rate of change is $\underline{\underline{\sqrt{17}}}$

in the direction $\underline{\langle 1, 4 \rangle}$.

5. Find the critical points of the function $f(x,y) = x^3 - 6x - y^2$ and classify them as maxima, minima, or saddle points. The graph might be helpful, but you can't rely on it alone -- you must *show* something is a saddle point, for instance, rather than just saying that it looks like one.



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$f(x,y) = x^3 - 6x - y^2$

$f_x = 3x^2 - 6$

$f_y = 2y$

To find critical points, set = to 0.

$3x^2 - 6 = 0$ $\qquad 2y = 0$

$3x^2 = 6$ $\qquad y = 0$

$x^2 = 2$

$x = \pm\sqrt{2}$

$\boxed{\text{Critical points} : (\pm\sqrt{2}, 0)}$

To find max, min, saddle pts:

$f_{xx} = 6x$

$f_{yy} = 2$

$f_{xy} = 0$

$D = f_{xx} f_{xy} - (f_{xy})^2$

Excellent

$x = \sqrt{2}$:

$D = 6\sqrt{2}(2) - 0$

$D = 12\sqrt{2}$ $\quad$ D is greater than 0, so it is either a max or a min.

Looking at f_{xx}:

$6\sqrt{2} > 0$ $\quad f_{xx} > 0$ so $(\sqrt{2}, 0)$ is a min.

$x = -\sqrt{2}$

$D = 6\,{}^{-}\!\sqrt{2}(2) - 0$

$D = -12\sqrt{2}$ $\quad$ D is less than zero, so $(-\sqrt{2}, 0)$ is a saddle point

$\boxed{(\sqrt{2},0) \text{ is a minimum}, \quad (-\sqrt{2},0) \text{ is a saddle point}}$

6. A group of puffins (puffins are sea birds, and while they're only distantly related to penguins some authorities have suggested that when unobserved they enjoy sliding on snow as much as penguins, and besides they have a funny name) has gone snowboarding. If a puffin is at the point (3,-5,2) on a surface shaped like the lower sheet of the hyperboloid $x^2 + y^2 - (z - 8)^2 = -2$, and the puffin's snowboard is of course tangent to the surface at that point, find an equation for the plane containing the snowboard.

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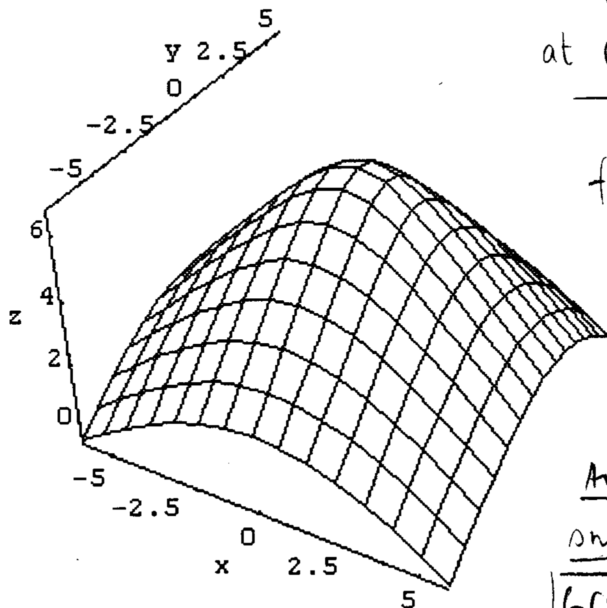


for a level curve $F(x,y,z) = k$, the tangent plane at (x_0, y_0, z_0) has equation

$$F_x(x_0,y_0,z_0)(x-x_0) + F_y(x_0,y_0,z_0)(y-y_0) + F_z(x_0,y_0,z_0)(z-z_0) = 0$$

$f(x,y,z) = x^2 + y^2 - (z-8)^2$

$f_x(x,y,z) = 2x \Rightarrow f_x(3,-5,2) = 2(3) = 6$

$f_y(x,y,z) = 2y \Rightarrow f_y(3,-5,2) = 2(-5) = -10$

$f_z(x,y,z) = -2(z-8) \Rightarrow f_z(3,-5,2) = -2(2-8) = 12$

An equation for the plane containing the snowboard is:

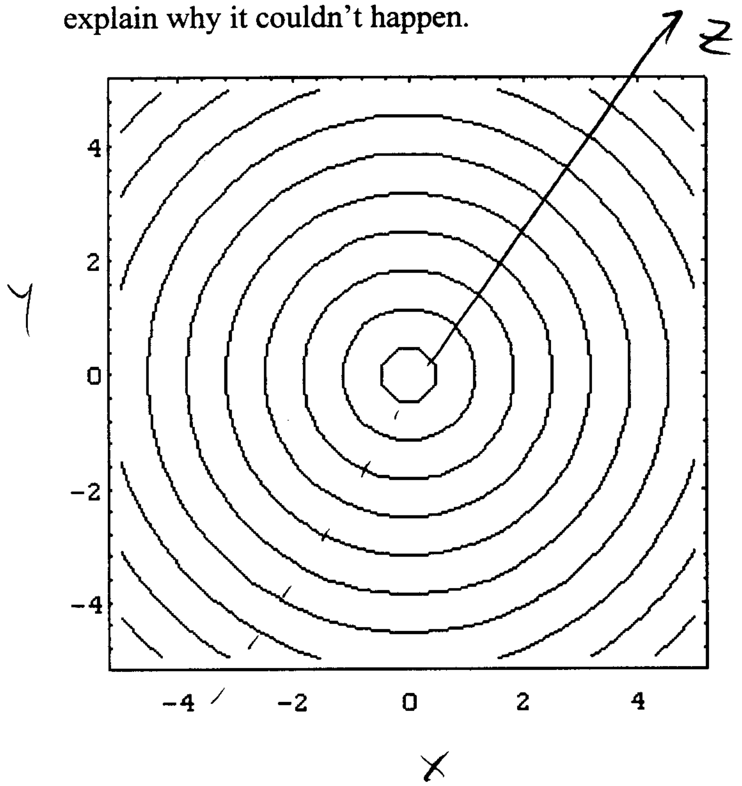
$$\boxed{6(x-3) - 10(y+5) + 12(z-2) = 0}$$

Excellent

7. Buffy is a student taking calculus at Oklahoma State University, and she's a little confused. She says "Oh my *God*. I'm, like, just *so* confused about these level curve thingys, you know? I mean, it's like, if I just know that all the level curves are circles or something, I don't think I can even, like, say what the graph thingy looks like, 'cause there could be, like, *totally* different graphs that have the same level curves. Like, this problem we had had a picture like this, but I think there are lots of different graphs that could be like that, you know?"

Is Buffy right? Either explain to her what different surfaces might have these level curves, or explain why it couldn't happen.

W



Since a height (or depth) of each contour line is not given, this could be a cone extending upwards on the z-axis or going downwards. Also, this is only a portion of a graph. It is possible for a graph to look like this at one point, but if you zoom out, you may find that the function oscillates (like cos()) so that there are many "dips" or "bumps" in the graph, as it were.

Good points.

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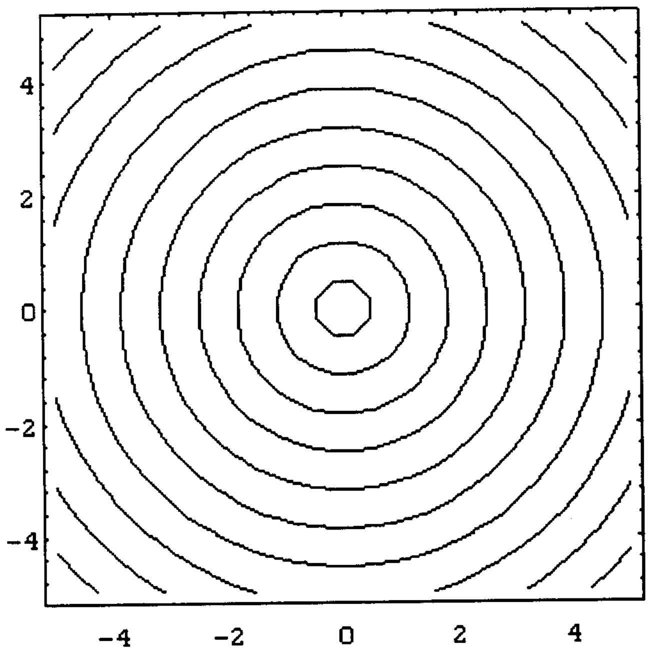

In the picture, since the z value at which the level curves were drawn is not indicated, it could be representing two possible graphs, one inverted from the other. In general, however, in a graph giving the z value for the level curves, a level curve drawing represents only one possible curve. It is also possible, nonetheless, that if the curves were drawn at the right intervals, some of the detail would be lost and it could be made to appear like another graph. For example, consider the graph $f(x,y)=\sqrt{x^2+y^2}$ This graph is a cone and could match the level curves above. Now consider the graph $\sqrt{x^2+y^2}\cos(\sqrt{x^2+y^2})$. If level curve were drawn at the right intervals, this graph could also match the level curves above. This would have to be done very deliberately, but it could be done.

Excellent points, and great example.

$$z - z_0 = \frac{\partial z}{\partial x}(x - x_0) + \frac{\partial z}{\partial y}(y - y_0)$$

8. Show that the z-intercept of the plane tangent to the paraboloid $z = x^2 + y^2$ at the point (x_0, y_0) is $z = -x_0^2 - y_0^2$.

$z_0 = x_0^2 + y_0^2$ Yes!

$\frac{\partial z}{\partial x} = 2x$ $\frac{\partial z}{\partial y} = 2y$

W

$z - z_0 = 2x_0(x - x_0) + 2y_0(y - y_0)$

$z - z_0 = 2xx_0 - 2x_0^2 + 2yy_0 - 2y_0^2$

z-intercept => $x + y = 0$ => $z - z_0 = 0 - 2x_0^2 + 0 - 2y_0^2$

$z - z_0 = -2x_0^2 - 2y_0^2$

$z - (x_0^2 + y_0^2) = -2x_0^2 - 2y_0^2$

$z = -\cancel{2}x_0^2 - \cancel{2}y_0^2 + \cancel{x_0^2} + \cancel{y_0^2}$

$\boxed{z = -x_0^2 - y_0^2}$

Very nice

10

9. Find an expression for a vector normal to a surface f(x,y) at the point (x_0,y_0). [You may assume that the surface is continuous at this point and that the partial derivatives both exist there. The point-normal equation for a plane $a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$ may be helpful.]

The plane tangent to a surface $f(x,y)$ at the point (x_0, y_0) is expressed by $z - z_0 = \frac{\partial z}{\partial x}(x - x_0) + \frac{\partial z}{\partial y}(y - y_0)$.

The vector normal to $f(x,y)$ at (x_0, y_0) will also be normal to the tangent plane expressed above.

According to the point-normal equation,

$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$.

Likewise, if we rearrange the tangent plane's equation into

$\frac{\partial z}{\partial x}(x - x_0) + \frac{\partial z}{\partial y}(y - y_0) - (z - z_0) = 0$,

we find that

$\langle \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1 \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$.

Well done

— This vector, then, is normal to both the tangent plane and the surface $f(x,y)$ at (x_0, y_0).

10. Find the maximum value of the plane f(x,y) = x + y subject to the constraint $x^2 + y^2 = 1$.

$\nabla f = \lambda \nabla G$ $\qquad$ $\nabla f = \langle 1, 1 \rangle$

let $f = x + y$ $\qquad$ $\nabla g = \langle 2x, 2y \rangle$

let $g = x^2 + y^2$

$\langle 1, 1 \rangle = \lambda \langle 2x, 2y \rangle$

$1 = 2x\lambda$ $\qquad$ $x = 1/2\lambda$

$1 = 2y\lambda$ $\qquad$ $y = 1/2\lambda$ $\quad \therefore x = y$ $\quad x^2 + x^2 = 1$

$x^2 + y^2 = 1$

$2x^2 = 1$

$x^2 = 1/2$

$x = \pm\sqrt{1/2}$ $\quad \therefore y = \pm\sqrt{1/2}$

$f(x,y) = x + y$

$f(x,y) = \sqrt{1/2} + \sqrt{1/2} = 2\sqrt{1/2}$ $\qquad$ $(\sqrt{1/2}, \sqrt{1/2})$

$f(x,y) = -\sqrt{1/2} + -\sqrt{1/2} = -2\sqrt{1/2}$ $\qquad$ $(-\sqrt{1/2}, -\sqrt{1/2})$

since $(\sqrt{1/2}, \sqrt{1/2})$ yeilds the highest valve of z, it the max. of the plane at $z = 2\sqrt{1/2}$

$f_x = 1$ $\quad f_{xx} = 0$ $\quad f_{xy} = 0$

$f_y = 1$ $\quad f_{yy} = 0$ $\quad D = (0)(0) - 0^2 = 0$?

Excellent

Extra Credit (5 points possible):

$+5$

Problem 10 finds the highest point on the trace of a particular plane f in the cylinder $x^2 + y^2 = 1$. Find a formula for the highest point on the trace of a general plane f in the cylinder $x^2 + y^2 = 1$. [Hint: At least try it for one other particular plane, say $f(x,y) = 2x + y$, then see if you can figure out how to generalize in this.]

$f(x,y) = 2x + y$, $g(x,y) = x^2 + y^2 = 1$

$\nabla f(x,y) = \langle 2, 1 \rangle$, $\nabla g(x,y) = \langle 2x, 2y \rangle$

$\nabla f = \lambda \nabla g$

$\Rightarrow 2 = 2x\lambda$

$1 = 2y\lambda$

$x^2 + y^2 = 1$

$\Rightarrow \lambda = \frac{2}{2x} = \frac{1}{x}$

$\Rightarrow 1 = \frac{2y}{x} \Rightarrow y = \frac{1}{2}x$

$\Rightarrow x^2 + \frac{1}{4}x^2 = 1$

$\Rightarrow \frac{5}{4}x^2 = 1$

$\Rightarrow x = \pm \frac{2\sqrt{5}}{5}$

$\Rightarrow y = \pm \frac{\sqrt{5}}{5}$

$\max. = f\left(\frac{2\sqrt{5}}{5}, \frac{\sqrt{5}}{5}\right) = 2\left(\frac{2\sqrt{5}}{5}\right) + \frac{\sqrt{5}}{5} = \sqrt{5}$

turn over

let's say we have a plane:

$$f(x,y) = ax + by + c$$

As before: $g(x,y) = x^2 + y^2 = 1$

$$\nabla f(x,y) = \langle a, b \rangle \quad , \quad \nabla g(x,y) = \langle 2x, 2y \rangle$$

$$\Rightarrow \nabla f = \lambda \nabla g$$

$$\Rightarrow \begin{aligned} a &= 2x\lambda \\ b &= 2y\lambda \\ x^2 + y^2 &= 1 \end{aligned}$$

$$\Rightarrow \lambda = \frac{a}{2x}$$

$$\Rightarrow b = \frac{2ay}{2x} = \frac{ay}{x}$$

$$\Rightarrow y = \frac{bx}{a}$$

$$\Rightarrow x^2 + \frac{b^2x^2}{a^2} = 1$$

$$\Rightarrow \frac{(a^2+b^2)x^2}{a^2} = 1$$

$$\Rightarrow x = \pm \frac{|a|}{\sqrt{a^2+b^2}}$$

$$\Rightarrow y = \frac{b}{a}\left(\pm \frac{|a|}{\sqrt{a^2+b^2}}\right) = \pm \frac{b}{\sqrt{a^2+b^2}}$$

$$\max = f\left(\frac{a}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}}\right) = a\left(\frac{a}{\sqrt{a^2+b^2}}\right) + b\left(\frac{b}{\sqrt{a^2+b^2}}\right) + c$$

$$= \frac{a^2+b^2}{\sqrt{a^2+b^2}} + C$$

$$= \sqrt{a^2+b^2} + C$$

Yes!