Calculus IV Exam 3 Spring 1999 4/22/99

Each problem is worth 10 points. Be sure to show all work for full credit. Please circle all answers and keep your work as legible as possible. Not responsible for damage due to clogs.

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1. Find $\int_C \langle y\sin x, -\cos x\rangle \cdot d\mathbf{r}$, where C is a <u>line segment</u> starting at $\overset{a}{(0,1)}$ and ending at $\overset{b}{(\pi,-1)}$.

$\frac{\partial P}{\partial y} = \sin x \qquad \frac{\partial Q}{\partial x} = \sin x$

They are equal so it is conservative : Fund. Theorem

$= f(r(b)) - f(r(a))$

find potential function:

$-y\cos x \qquad -y\cos x$

$f = -y\cos x$

Plug in a and b:

$(1\cos\pi) + (1\cos 0)$

$-1 + 1 = \boxed{0}$

Perfect

2. Compute $\int_C (x^2+y^2)dx - x\,dy$ along the counterclockwise quarter circle from (1,0) to (0,1).

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conservative? $2y \neq -xy$ No. Not closed either.

Use Long way! ;)

$\vec{F} = \langle x^2+y^2, -x \rangle$

I. $x(t) = \cos t$

$y(t) = \sin t$

$\vec{r} = \langle \cos t, \sin t \rangle$

Great

II. $\vec{F}(\vec{r}) = \langle \cos^2 t + \sin^2 t, -\cos t \rangle = \langle 1, -\cos t \rangle$

III. $\vec{r}\,' = \langle -\sin t, \cos t \rangle$

IV $\int_0^{\pi/2} \langle 1, -\cos t \rangle \cdot \langle -\sin t, \cos t \rangle \, dt = \int_0^{\pi/2} (-\sin t - \cos^2 t)\,dt$

$= \int_0^{\pi/2} \left(-\sin t - \frac{1}{2} - \frac{1}{2}\cos 2t\right)dt = \left[\cos t - \frac{1}{2}t - \frac{1}{4}\sin 2t\right]_0^{\pi/2}$

$= \left(\cos\frac{\pi}{2} - \frac{\pi}{4} - \frac{1}{4}\sin\pi\right) - \left(\cos 0 - \frac{1}{4}\sin 0\right)$

$= \left(0 - \frac{\pi}{4} - 0\right) - (1 - 0) = \boxed{-\frac{\pi}{4} - 1}$

3. Compute $\oint y^2\,dx + xy\,dy$ for the path C consisting of the first-quadrant portion of a circle (centered at the origin) of radius 3 traversed counterclockwise, along with the line segments from (0,3) to (0,0) and from (0,0) to (3,0).

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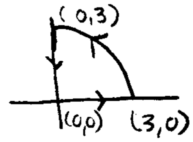
$x^2+y^2=9$ (circle of radius 3)



simple closed path
$\Rightarrow$ Green's Theorem

$Q = xy \qquad P = y^2$

$\partial Q/\partial x = y \qquad \partial P/\partial y = 2y$

$$\iint_D \left(\partial Q/\partial x - \partial P/\partial y\right) dA = \int_C P\,dx + Q\,dy$$

$$\int\int (y-2y)\,dA = \int\int -y\,dA = \int_0^{\frac{\pi}{2}}\int_0^3 -r\sin\theta\, r\,dr\,d\theta$$

$$\int_0^{\frac{\pi}{2}} -\tfrac{1}{3}r^3\sin\theta\,\Big]_0^3\,d\theta = \int_0^{\frac{\pi}{2}} \left(-\tfrac{1}{3}\right)(3^3)\sin\theta\,d\theta = \int_0^{\frac{\pi}{2}} -9\sin\theta\,d\theta$$

$$+9\cos\theta\,\Big]_0^{\frac{\pi}{2}} = 9\cos\underbrace{\tfrac{\pi}{2}}_{0} - 9\cos\underbrace{0}_{1} = \underline{\underline{-9}}$$

Very nice

4. **The Earth constantly radiates heat into space, partly due to internal cooling and partly due to reflected energy from the Sun. Suppose this heat is radiated according to the vector field F(x,y,z) = 10xi +10yj + zk. What is the flux of this vector field through the surface of the Earth's upper atmosphere, which forms a sphere about 7300 kilometers in radius?**

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$\vec{F}(x,y,z) = \langle 10x, 10y, z \rangle$

S is a closed surface $\Rightarrow$ by Divergence Theorem, the flux is $\iint_S \vec{F} \cdot d\vec{s} = \iiint_E \operatorname{div} \vec{F} \, dv$

$\operatorname{div} \vec{F} = 10 + 10 + 1 = 21$ $\Rightarrow$ the flux is $\iiint_{Earth} 21 \cdot dv$

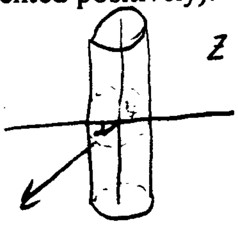
$= 21 \iiint_{Earth} 1 \, dv = 21V$ Great, where V is the volume of the Earth

$\boxed{\text{Flux} = 21 \cdot \frac{4\pi R^3}{3} = 28\pi R^3}$ where R = Radius of the Earth

I am sure you don't wanna how large this is

But just in case it comes to about 3.42×10^{13} units

5. When water drains out of a bathtub and swirls down a pipe, the vector field representing the velocity of the water might be modeled by $\mathbf{F}(x,y,z) = -y\mathbf{i} + x\mathbf{j} - 5\mathbf{k}$. Suppose the pipe is a cylinder of radius of 2 cm (centered around the z axis) and has a filter screen shaped like the plane $2x + 3y + z + 6 = 0$ in it. Compute the flux of this vector field through this surface (assume the surface is oriented positively). $2x+3y+z+6=0$



$z = -2x - 3y - 6$

$3y = -2x - 6$

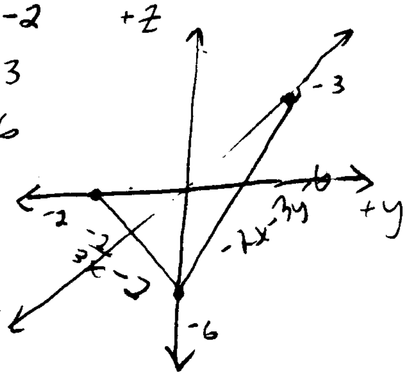
$y = \frac{-2}{3}x - 2$

$x = -3$

when $z=0$ and $x=0$, $y=-2$

" " $z=0$ and $y=0$, $x=-3$

" " $y=0$ and $x=0$, $z=-6$



Curl $\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & -5 \end{vmatrix} = (0-0)\hat{i} + (0-0)\hat{j} + (1-(-1))\hat{k}$

$= -2\hat{k}$, $\therefore$ does not have a potential.

[redacted]

$\iint_D \vec{F}(\vec{r}(u,v)) \cdot [\vec{r}_u \times \vec{r}_v]\, dA$ $\quad z = -2x - 3y - 6$ $\quad x(u,v) = u$, $y(u,v) = v$, $z(u,v) = -2u - 3v - 6$

[redacted]

1. [redacted]

[redacted]. $\vec{r}_u = \langle 1, 0, -2 \rangle$ $\quad \vec{r}_v = \langle 0, 1, -3 \rangle$

$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -2 \\ 0 & 1 & -3 \end{vmatrix} = (0-(-2))\hat{i} + (0-(-3))\hat{j} + (1-0)\hat{k} = 2\hat{i} + 3\hat{j} + \hat{k}$

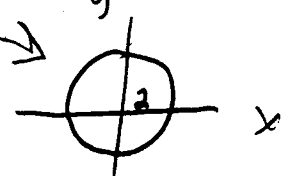
[redacted]

[redacted]. $\iint_D \langle -u, v, -5 \rangle \cdot \langle 2, 3, 1 \rangle\, dA$

$= \iint_D (-2u + 3v - 5)\, dA$ $\quad$ Now for some hard limit setup.

u is acting as x and v is y

Top View



$= \int_0^{2\pi} \int_0^2 (-2r\cos\theta + 3r\sin\theta - 5)\, r\, dr\, d\theta.$ $\quad$ Now let's use polar.

$= \int_0^{2\pi} \left[-\frac{2}{3}r^3\cos\theta + r^3\sin\theta - \frac{5}{2}r^2 \right]_0^2 d\theta = \int_0^{2\pi} \left(\frac{-16}{3}\cos\theta + 8\sin\theta - 10 \right) d\theta$

$= \left[\frac{-16}{3}\sin\theta - 8\cos\theta - 10\theta \right]_0^{2\pi} = (0 - 8 - 20\pi) - (0 - 8 - 0)$

$= \cancel{-8} - 20\pi \cancel{+8} =$ [redacted]

Very nice

W

6. Show that for any vector field **F**(x,y,z) = P(x,y,z)**i** + Q(x,y,z)**j** + R(x,y,z)**k**, so long as P, Q, and R have continuous second-order partial derivatives, div(curl **F**) = 0. How is the requirement that the partials be continuous necessary?

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$\vec{F} = \langle P, Q, R \rangle \qquad \nabla \cdot (\nabla \times F) = 0$

1st

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}$$

$$\nabla \times \vec{F} = \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle$$

now,

$$\nabla \cdot (\nabla \times \vec{F}) = \left[\frac{\partial}{\partial x}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\right] + \left[\frac{\partial}{\partial y}\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\right] + \left[\frac{\partial}{\partial z}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\right]$$

$$= \left(\cancel{\frac{\partial^2 R}{\partial x \partial y}} - \cancel{\frac{\partial^2 Q}{\partial z \partial x}}\right) + \left(\cancel{\frac{\partial^2 P}{\partial y \partial z}} - \cancel{\frac{\partial^2 R}{\partial x \partial y}}\right) + \left(\cancel{\frac{\partial^2 Q}{\partial z \partial x}} - \cancel{\frac{\partial^2 P}{\partial y \partial z}}\right) = 0$$

It is important that the partials be continous so that you can switch the order of differentiation for instance

$$\frac{\partial^2 Q}{\partial x \partial z} \text{ to } \frac{\partial^2 \partial Q}{\partial z \partial z},$$

Great

7. Buffy is having trouble with conservative vector fields. "Oh my *God*, I just, like, don't *get* it. I mean, like, we looked at this one, you know? And I, like, told this guy Biff that I was studying with that it was, like, conserva-whatever 'cause when you, like, draw a half circle going, like, either way around from up at the top to down at the bottom, it's like, totally across the arrow thingys so it's like zero either way, y'know? So I think it's conserva-whatever, but this Biff guy says it's not, but, y'know, he's not very smart, I think maybe, so I dunno, y'know?"

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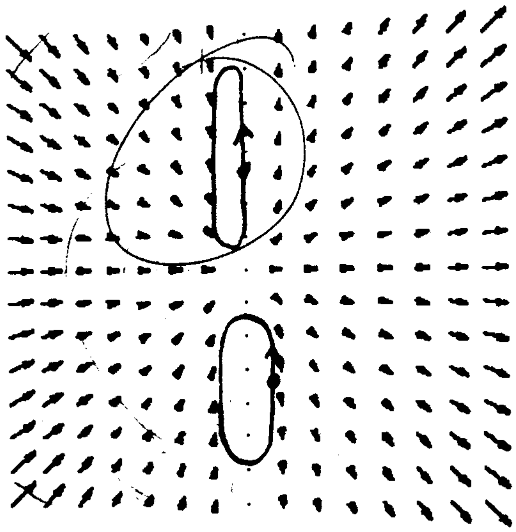
Who's right, Biff or Buffy, and what do you think of her reasoning?



Biff is Right.

She's okay in here reasoning... the only problem is that she didn't take is far enough. It appears to me that each (the right half & the left half) are conservative, but when you put the halves together. you get something wicked going on at the vertical centerline.

If she draws a circles like the circles I just drew, she'll discover that it's not a conservative field.

In a conservative field, if the path is simple & closed, it's work is 0.

Good

8. Show that in any vector field **F**(x,y) = P(x,y)**i** + 0**j**, where P has continuous partial derivatives, the line integral through **F** along any vertical line segment C will be zero (this is essentially the portion of the proof of Green's Theorem that I hand-waved in class). [Hint: If you have trouble figuring how to do it in general, warm up with your favorite particular vertical line segment and a nice vector field like **F**(x,y) = xy**i** + 0**j**, then see if you can generalize.]

$(1,0)$ to $(1,2)$

→ not conservative → not closed → long way

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I. $x(t) = 1$
$y(t) = 2t$
$\vec{r}(t) = \langle 1, 2t \rangle$
$0 \le t \le 1$

II. $\vec{F}(\vec{r}) = \langle 2t, 0 \rangle$

III. $\vec{r}\,'(t) = \langle 0, 2t \rangle$

IV. $\int_C \vec{F}(\vec{r}) \cdot \vec{r}\,'(t)\,dt = \int_C \langle 2t, 0 \rangle \cdot \langle 0, 2t \rangle \, dt = 0$

... more generally, on a vertical segment (a, b) to (a, c) in the field $\vec{F} = P\vec{i} + 0\vec{j}$,

I. $x(t) = a$
$y(t) = b + (c-b)t$
$\vec{r}(t) = \langle a, b + (c-b)t \rangle$
$0 \le t \le 1$

II. $\vec{F}(\vec{r}) = \langle P(a, b+(c-b)t), 0 \rangle$

III. $\vec{r}\,'(t) = \langle 0, c-b \rangle$

IV. $\int_C \vec{F}(\vec{r}) \cdot \vec{r}\,'(t)\,dt = \int_C \langle P(a, b+(c-b)t), 0 \rangle \cdot \langle 0, c-b \rangle \, dt$

$= 0$ NO MATTER WHAT!

The values of a, b, and c make no difference
The function P(x, y) makes no difference.

Excellent

9. The parametric equations x = cosh u cos v, y = cosh u sin v, z = sinh u, represent a hyperboloid of one sheet. Show that the surface area of such a hyperboloid between the planes z = a and z = b is given by the integral $\int_0^{2\pi}\int_{\sinh^{-1}a}^{\sinh^{-1}b} \cosh u\sqrt{\cosh^2 u+\sinh^2 u}\,du\,dv$. [Hint: Recall that (sinh x)′ = cosh x and (cosh x)′ = sinh x.]

12

$a = \sinh u;\ u = \sinh^{-1} a$

$b = \sinh u;\ u = \sinh^{-1} b$

$\cos v, \sin v$ +2

$0 \le v \le 2\pi$

"one full sweep."

$\vec{r} = \langle \cosh u \cos v, \cosh u \sin v, \sinh u \rangle$

$\vec{r}_u = \langle \sinh u \cos v, \sinh u \sin v, \cosh u \rangle$

$\vec{r}_v = \langle -\cosh u \sin v, \cosh u \cos v, 0 \rangle$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \sinh u \cos v & \sinh u \sin v & \cosh u \\ -\cosh u \sin v & \cosh u \cos v & 0 \end{vmatrix}$$

$$= (-\cosh^2 u \cos v)\hat{i} + (-\cosh^2 u \sin v)\hat{j} + (\sinh u \cosh u \cos^2 v + \sinh u \cosh u \sin^2 v)\hat{k}$$

$$= |(\ \text{"}\ )\hat{i} + (\ \text{"}\ )\hat{j} + (\sinh u \cosh u)\hat{k}|$$

$$= \sqrt{\cosh^4 u \cos^2 v + \cosh^4 u \sin^2 v + \sinh^2 u \cosh^2 u}$$

$$= \sqrt{\cosh^4 u + \sinh^2 u \cosh^2 u}$$

$$= \cosh u \sqrt{\cosh^2 u + \sinh^2 u}$$

$$\boxed{\int_0^{2\pi}\int_{\sinh^{-1}a}^{\sinh^{-1}b} \cosh u \sqrt{\cosh^2 u + \sinh^2 u}\,du\,dv}$$

Very nice

10. Stewart gives a simpler formula for flux integrals through a vector field **F** = **Pi** + **Qj** +**Rk** when the surface involved can be expressed in the form z = g(x,y). Show that in this case the general version we used reduces to $\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R \right) dA$.

W

$\vec{r} = \langle x, y, g \rangle$

$\vec{r}_x = \langle 1, 0, \frac{\partial g}{\partial x} \rangle$

$\vec{r}_y = \langle 0, 1, \frac{\partial g}{\partial y} \rangle$

$\iint_D \vec{F}(\vec{r}) \cdot [\vec{r}_u \times \vec{r}_v] \, dA$ if $\vec{F} = \langle P, Q, R \rangle$

then $\rightarrow$ $[\vec{r}_u \times \vec{r}_v] = \langle -\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1 \rangle$

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & \frac{\partial g}{\partial x} \\ 0 & 1 & \frac{\partial g}{\partial y} \end{vmatrix} = \left(0 - \frac{\partial g}{\partial x}\right)\vec{i} + \left(0 - \frac{\partial g}{\partial y}\right)\vec{j} + (1-0)\vec{k}$$

$$\vec{r}_x \times \vec{r}_y = \langle -\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1 \rangle$$

$$\iint_D \vec{F} \cdot [\vec{r}_x \times \vec{r}_y] \, dA = \iint_D \langle P, Q, R \rangle \cdot \langle -\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1 \rangle \, dA$$

$$= \boxed{\iint_D -P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R \, dA}$$

Great

plane: $ax+by+cz+d=0$ radius $= R$

I. $z = -\frac{a}{c}x - \frac{b}{c}y - \frac{d}{c}$

$x(u,v) = u$
$y(u,v) = v$
$z(u,v) = -\frac{a}{c}u - \frac{b}{c}v - \frac{d}{c}$

$\vec{r} = \langle u, v, -\frac{a}{c}u - \frac{b}{c}v - \frac{d}{c} \rangle$

II. $\vec{F}(\vec{r}) = \langle -v, u, -5 \rangle$

III. $\vec{r}_u = \langle 1, 0, -\frac{a}{c} \rangle$
$\vec{r}_v = \langle 0, 1, -\frac{b}{c} \rangle$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -\frac{a}{c} \\ 0 & 1 & -\frac{b}{c} \end{vmatrix} = \left\langle \frac{a}{c}, \frac{b}{c}, 1 \right\rangle$$

IV $$\iint_D \langle -v, u, -5 \rangle \cdot \left\langle \frac{a}{c}, \frac{b}{c}, 1 \right\rangle dA = \iint_D \left(\frac{b}{c}u - \frac{a}{c}v - 5 \right) dA$$

$$= \int_0^{2\pi} \int_0^R \left(\frac{b}{c} r\cos\theta - \frac{a}{c} r\sin\theta - 5 \right) r \, dr \, d\theta = \int_0^{2\pi} \int_0^R \left(\frac{b}{c} r^2 \cos\theta - \frac{a}{c} r^2 \sin\theta - 5r \right) dr \, d\theta$$

$$= \int_0^{2\pi} \left(\frac{b}{3c} r^3 \cos\theta - \frac{a}{3c} r^3 \sin\theta - \frac{5}{2} r^2 \right) \Big|_0^R d\theta = \int_0^{2\pi} \left(\frac{b}{3c} R^3 \cos\theta - \frac{a}{3c} R^3 \sin\theta - \frac{5}{2} R^2 \right) d\theta$$

$$= \left(\frac{b}{3c} R^3 \sin\theta + \frac{a}{3c} R^3 \cos\theta - \frac{5}{2} R^2 \theta \right) \Big|_0^{2\pi} = \left(0 + \cancel{\frac{a}{3c} R^3} - \frac{10\pi}{2} R^2 \right) - \left(0 + \cancel{\frac{a}{3c} R^3} - 0 \right)$$

$$= \frac{-10\pi}{2} R^2$$

The flux obviously only depends on the radius of the pipe and not on any aspect of the plane. The flux magnitude becomes greater proportionately with the square of the radius of the pipe. The value will always be negative, which make sense since the water flows down while the plane is oriented up.

Absolutely Great