Exam 2 Calc 1 7/15/2004

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. Prove the Sum Rule for derivatives.

$$[f(x)+g(x)]' = \lim_{h \to 0} \frac{[f(x+h)+g(x+h)] - [f(x)+g(x)]}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h)-f(x)+g(x+h)-g(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h}$$

$$= f'(x) + g'(x) \quad \square$$

Great

2. Find the exact x values where the absolute minimum and maximum values of the function $f(x) = x^3 - 6x + 5$ occur on the interval [0, 2].

$f'(x) = 3x^2 - 6$

$3x^2 - 6 = 0$

$+6 \quad +6$

$\frac{3x^2}{3} = \frac{6}{3}$

$\sqrt{x^2} = \sqrt{2}$

$x = \pm\sqrt{2}$

Took out $-\sqrt{2}$ because not in range

$f''(x) = 6x$

$= 6(\sqrt{2})$

$= 8.485$

Since + will have a minimum

Possibility for points:

$0, \sqrt{2}, 2$

$f(0) = (0)^3 - 6(0) + 5 = 5$

$f(\sqrt{2}) = (\sqrt{2})^3 - 6(\sqrt{2}) + 5 = -.656$

$f(2) = (2)^3 - 6(2) + 5 = 1$

So,

$(0, 5)$ is the absolute maximum

$(\sqrt{2}, -.656)$ is the absolute minimum

Excellent

3. A curve is given parametrically by $x(t) = t - t^2 - 2$, $y(t) = t^2 + t + 2$.
   a) When $t = 1$, what are the coordinates of the corresponding point?

   b) Find the coordinates of all points on the curve where the tangent line is horizontal.

$x(1) = (1) - (1)^2 - 2 = -2$

a) $y(1) = (1)^2 + (1) + 2 = 4$ So when $t = 1$, the corresponding point is $(-2, 4)$

b) $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t+1}{1-2t}$, so the tangent is horizontal when this is zero, which will happen when the numerator is zero, or $2t+1=0$ $\Rightarrow 2t = -1 \Rightarrow t = -1/2$. But the question asked for a points, so when $t = -1/2$:

$x(-1/2) = (-1/2) - (-1/2)^2 - 2 = -2\,3/4$

$y(-1/2) = (-1/2)^2 + (-1/2) + 2 = 1\,3/4$ $(-2\,3/4, 1\,3/4)$

4. If $f(2) = 5$, $f'(2) = 1$, $g(2) = -3$, and $g'(2) = 7$, what are:
   a) $(f - g)'(2)$? $f'(2) - g'(2) = 1 - 7 = \boxed{-6}$

   b) $(f \times g)'(2)$? $f'(2) \cdot g(2) + f(2) \cdot g'(2)$ $\Rightarrow (1 \cdot -3) + (5 \cdot 7)$ $\Rightarrow \boxed{32}$

   c) $\left(\frac{f}{g}\right)'(2)$? $\frac{[f'(2) \cdot g(2)] - [f(2) \cdot g'(2)]}{[g(2)]^2} = \frac{(1 \cdot -3) - (5 \cdot 7)}{(-3)^2} = \boxed{-\frac{38}{9}}$

Great

5. Find $\lim_{x \to 0} \frac{\sin(3x)}{x}$. Be clear in your justification!

Since both the numerator and denominator are approaching 0 L'Hopital's Rule can be used.

$$\lim_{x \to 0} \frac{\sin 3x}{x} \overset{LH}{=} \lim_{x \to 0} \frac{3\cos 3x}{1}$$

Plugging in zero for x shows that as x approaches zero $\frac{3\cos 3x}{1}$ will be approaching the value of 3.

$$\lim_{x \to 0} \frac{3\cos 3(0)}{1} = \boxed{3}$$

Well done.

6. Prove the Quotient Rule for derivatives [you may feel free to use the Product Rule if you like].

$\left[\frac{f(x)}{g(x)}\right]' = \left[f(x) \cdot \frac{1}{g(x)}\right]'$ First the reciprical rule needs to be proven.

Proof: $\left(\frac{1}{g(x)}\right)' = \lim_{h \to 0} \frac{\frac{1}{g(x+h)} - \frac{1}{g(x)}}{h} = \lim_{h \to 0} \frac{g(x) - g(x+h)}{g(x+h)g(x)} \cdot \frac{1}{h}$

$= \lim_{h \to 0} \frac{-1(g(x+h) - g(x))}{h\, g(x+h)g(x)} = \lim_{h \to 0} \frac{-1}{g(x+h)g(x)} \cdot \frac{g(x+h) - g(x)}{h}$ ← definition of $g'(x)$

$= \lim_{h \to 0} \left(\frac{-g'(x)}{g(x+h)g(x)}\right) = \frac{-g'(x)}{[g(x)]^2}$

Using the product rule then: ← from reciprical rule

$$\left[f(x) \cdot \frac{1}{g(x)}\right]' = f'(x)\left[\frac{1}{g(x)}\right] + f(x)\left(\frac{-g'(x)}{[g(x)]^2}\right)$$

$$= \frac{g(x)}{g(x)} \cdot \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{[g(x)]^2}$$

$$\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2} \quad \square$$

Nice Job!

7. Find all points on the curve defined by the equation $y^2 + 2xy + x^2 + x - y = 0$ where the tangent line is vertical.

I. Differentiate implicitly: $2y \cdot y' + 2y + 2x \cdot y' + 2x + 1 - y' = 0$

$$2y\,y' + 2x\,y' - y' = -2y - 2x - 1$$

$$y'(2y + 2x - 1) = -2y - 2x - 1$$

$$y' = \frac{-2y - 2x - 1}{2y + 2x - 1}$$

II. So vertical tangent where denominator of y' is zero:

$$2y + 2x - 1 = 0$$

$$2x = 1 - 2y$$

$$x = \tfrac{1}{2} - y$$

III. Put back in original equation to find a point on the curve satisfying this:

$$y^2 + 2\left(\tfrac{1}{2} - y\right)y + \left(\tfrac{1}{2} - y\right)^2 + \left(\tfrac{1}{2} - y\right) - y = 0$$

$$y^2 + y - 2y^2 + \tfrac{1}{4} - \tfrac{1}{2}y - \tfrac{1}{2}y + y^2 + \tfrac{1}{2} - y - y = 0$$

$$-2y + 3/4 = 0$$

$$y = 3/8$$

IV. Plug this back into our equation from part II to find the x-coordinate:

$$x = \tfrac{1}{2} - \left(\tfrac{3}{8}\right)$$

$$x = \tfrac{1}{8}$$

So $\left(\tfrac{1}{8}, \tfrac{3}{8}\right)$ is the only point where the tangent is vertical.

8. Bunny is a calculus student at a large state university and she's having some trouble. She says "So this second derivative test thingy totally confuses me. First, I can't get it straight what it means when it's plus or minus, 'cause it seems kinda backwards or something. But I'm more confused than just that, too, 'cause I don't think it makes sense anyway. I mean, if there was a spot on the graph where it was a minimum, like the shape at the bottom of a parabola or whatever, then wouldn't *all* the derivatives be zero, like the first derivative and the second derivative too, and all the rest? Because it's not changing there, right, like it's not up or down at all? So *all* the derivatives would be zero and they wouldn't have plus or minuses at all? And I think I better figure this out, or else my grade might not be good enough to pledge a good sorority!"

Explain clearly to Bunny what's up with the second derivative at a local maximum or minimum point, and whether it can (or must) be zero at such a point.

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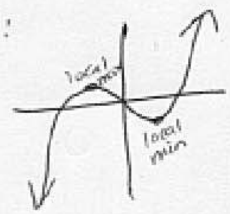
First, we use the first derivative to find critical points, which are local (at least local, depending on boundaries and what-not) minimums or maximums, ~~[illegible]~~.

* only use critical points from f'(x) in the second derivative!

The second derivative is used to determine whether the points we found are local maximums or minimums, ~~[illegible]~~. If we put in an x-value ~~point~~ we found ~~and~~ into the second derivative and the output is a negative number, that is at the very least a local maximum. Similarly, if the x-value ~~number~~ we put into the second derivative yields a positive number, that is at the very least a local minimum.

Excellent answer.

now, if there are no boundaries, these are definitely local maximums & minimums. for example:



But if the function is restricted to a certain number range, the critical point x-values and the edges of the boundaries need to be entered into the original equation to find which yield the greatest and least y-values, in effect the absolute maximums and minimums.

9. Find a function of the form $f(x) = ax^2 + bx + c$ (for some values of the constants a, b, and c) which passes through both the origin and the point (2,3) and has a tangent at (2,3) with a slope of 5.

If it passes through the origin, then there is no c-value because f(0) would not equal 0.

Yes!

we also know that $a(2)^2 + b(2) = 3$

$4a + 2b = 3$

b/c the pt (2,3) is on the parabola. Great!

If the tangent has a slope of 5 at the point (2,3), then the derivative must equal 5 when $x=2$

$f(x) = ax^2 + bx$

$f'(x) = 2ax + b$

$2ax + b = 5$

$2a(2) + b = 5$

$4a + b = 5$

$b = 5 - 4a \xrightarrow{\text{substitute}}$

$4a + 2b = 3$

$4a + 2(5-4a) = 3$

$4a + 10 - 8a = 3$

$-4a = -7$

$a = \frac{7}{4}$

$b = 5 - 4\left(\frac{7}{4}\right) \xleftarrow{\text{substitute}}$

$b = -2$

Outstanding!

$\therefore f(x) = \frac{7}{4}x^2 - 2x$

10. Suppose that f(x) is a function with f(x) > 0 for all values of x. Let g(x) = 1/f(x).

a) If f has a local maximum at x_1, what can you say about g? How do you know?

b) If f is concave up at x_2, what can you say about g? How do you know?

Extra Credit (5 points possible):

What can you say about the derivative of arcsinh x?

If f has a local max at x_1, probably* $f'(x_1)=0$ and $f''(x_1)<0$.

Now $g'(x) = \left(\frac{1}{f(x)}\right)' = \frac{0\cdot f(x) - 1\cdot f'(x)}{f^2(x)} = -\frac{f'(x)}{f^2(x)}$

so $g'(x_1) = -\frac{f'(x_1)}{f^2(x_1)} = -\frac{0}{f^2(x_1)} = 0$ (as long as $f(x_1)\neq 0$)

And $g''(x) = \left(-\frac{f'(x)}{f^2(x)}\right)' = -\frac{f''(x)\cdot f^2(x) - f'(x)\cdot 2f(x)\cdot f'(x)}{[f^2(x)]^2}$

$= \frac{2[f'(x)]^2 f(x) - f''(x)\cdot[f(x)]^2}{[f(x)]^4}$

So $g''(x_1) = \frac{2[0]^2 f(x_1) - f''(x_1)[f(x_1)]^2}{[f(x_1)]^4} > 0$ since $f''(x_1)<0$.

a) So at x_1, $g'=0$ and $g''>0$ and that means a local minimum (again as long as $f(x_1)\neq 0$)

b) From the above, g'' has the opposite sign of f'', so g would be concave down.

* "probably" because if the second derivative test fails we might have a local maximum where $f''(x_1)=0$, in which case the second derivative test fails on g as well. The same conclusions would still hold, but justifying why would be much harder in this case.