Exam 3 Calc 1 7/21/2005

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. Find an antiderivative for the function $f(x) = 12x^2 - 3$ for which $F(1) = 2$.

$f(x) = 12x^2 - 3$

$F(x) = 4x^3 - 3x + C$

$2 = 4(1)^3 - 3(1) + C$

$2 = 4 - 3 + C$

$2 = 1 + C$

$-1 \quad -1$

$1 = C$

Good

$F(x) = 4x^3 - 3x + 1$

2. Find the interval(s) of increase of the function $g(x) = 2x^3 - 5x^2 + 4$.

$g'(x) = 6x^2 - 10x$

$0 = 6x^2 - 10x$

$= 2x(3x - 5)$

$x = 0$ & $x = \frac{5}{3}$ changes whether increasing or decreasing

$g'(-1) = 6(-1)^2 - 10(-1)$

$= 16$

Plug in value less than 0

increasing since positive

$\boxed{\text{increasing: } (-\infty, 0)\ (\tfrac{5}{3}, \infty) \quad \text{decreasing: } (0, \tfrac{5}{3})}$

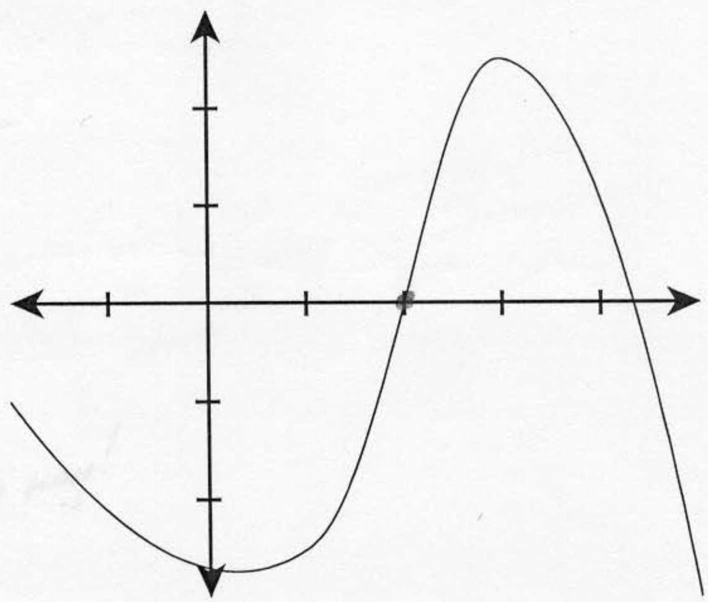
since no double roots, It can be said that it changes whether increasing or decreasing at $x = 0$ & $x = \frac{5}{3}$

Great

3. Use the graph of the function shown below to answer the following questions, assuming that beyond the region shown the graph continues to follow the trends of the portion shown. Assume that the tick marks along the axes are at unit intervals, and approximate values as necessary.

a) State the interval(s) on which the function is concave down.

$(2, \infty)$



b) State the approximate coordinates of any inflection point(s) of the function.

$(2, 0)$ Yes

4. Use calculus to find the absolute minimum and absolute maximum values of

$$f(x) = 2x^3 - 3x^2 - 12x + 1$$

on the interval [-3,3].

first, plug in boundaries of $x=-3$ & $x=3$

$f(3) = 2(3)^3 - 3(3)^2 - 12(3) + 1$
$= -8$

$f(-3) = 2(-3)^3 - 3(-3)^2 - 12(-3) + 1$
$= -44$

derive for local max/min(s)

$f'(x) = 6x^2 - 6x - 12$
$0 = 6(x^2 - x - 2)$
$0 = (x-2)(x+1)$
$x = -2$
$x = -1$

plug in to original

$f(-2) = 2(-2)^3 - 3(-2)^2 - 12(-2$
$= -3$

$f(-1) = 2(-1)^3 - 3(-1)^2 - 12(-1) + 1$
$= 8$

$\boxed{\text{max: } 8 \quad \text{min: } -44}$

Excellent!

5. Use Newton's Method with the initial approximation $x_1 = 1$ to find x_3, the third approximation to the root of the equation $x^3 + 2x - 5 = 0$.

$$x^3 + 2x - 5 = 0$$
$$3x^2 + 2 = 0$$

$$1^3 + 2 - 5 = -2$$
$$3 - 5 = -2$$
$$3 + 2 = 5$$

$$x_2 = 1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_2 = \frac{5}{5} - \frac{-2}{5} = \frac{7}{5}$$

$$\left(\frac{7}{5}\right)^3 + 2\cdot\frac{7}{5} - 5$$

$$\frac{343}{125} + \frac{14}{5} - 5$$

$$\frac{343}{125} + \frac{350}{125}$$

$$x_3 = \frac{7}{5} - \frac{\frac{68}{125}}{\frac{197}{25}}$$

$$3\cdot\left(\frac{7}{5}\right)^2 + 2$$

$$x_3 = \frac{175}{125} - \frac{\frac{68}{125}}{\frac{985}{125}}$$

$$\frac{693}{125} - \frac{625}{125} \quad \frac{68}{125}$$

$$3 \cdot \frac{49}{25} = \frac{147}{25} + 2$$

$$\frac{197}{25}$$

$$x_3 = 1.330964$$

Excellent

6. Evaluate $\lim_{x\to 0^+} \sqrt{x}\ln x$.

Both numerator and denominator go to $-\infty$, so use L'Hôpital's.

$$\lim_{x\to 0^+} \frac{\ln x}{x^{-1/2}} \overset{L'H}{=} \lim_{x\to 0^+} \frac{1/x}{-\frac{1}{2}x^{-3/2}} = \lim_{x\to 0^+} -2x^{-1}\cdot x^{3/2} = \lim_{x\to 0^+} -2x^{1/2}$$

$$= 0$$

7. Find the point on the line $5x + 3y = 15$ that lies closest to the point (1,0).

$5x_1 + 3y_1 = 15$

$3y_1 = 15 - 5x_1$

$y_1 = 5 - \frac{5}{3}x_1$

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$= \sqrt{(1 - x_1)^2 + (0 - y_1)^2}$

$= \sqrt{(1 - x_1)^2 + y_1^2}$

substitute $d^2 = (1 - x_1)^2 + (\frac{5}{3}x_1 - 5)^2$

$= 1 - 2x_1 + x_1^2 + \frac{25}{9}x_1^2 - \frac{50}{3}x_1 + 25$

$d^2 = \frac{34}{9}x_1^2 - \frac{56}{3}x_1 + 26$

$(d^2)' = \frac{68}{9}x - \frac{56}{3}$

$0 = \frac{68}{9}x - \frac{56}{3}$

$\boxed{x_1 = \frac{42}{17}}$ ⇒ has to be minimum (no longest distance)

$\Leftarrow d^2$ has same x values for max/min as does d

Plug in to find y

$5(\frac{42}{17}) + 3y = 15$

$\frac{210}{17} + 3y = 15$

$3y = \frac{45}{17}$

$\boxed{y_1 = \frac{15}{17}}$

$\boxed{\left(\frac{42}{17}, \frac{15}{17}\right)} \approx (2.471, .882)$

Wonderful!

8. Bunny is a calculus student at Enormous State University, and she's wondering about something. Bunny says "So we just learned about these antiderivative thingies, and I was wondering about them. Like, if you had something like x times e to the x, and you wanted its antiderivative, would you use the Product Rule on it just like you would with derivatives? I mean, like, do the antiderivative of the first times the second, plus the first times the antiderivative of the second? Or is that only for derivatives?"

Help Bunny out by **explaining** how she could tell, and whether it works the way she describes.

Well Bunny, the nice thing about antiderivatives is that you can always check them. If what you're suggesting is true, then the function it produces (which would be $\frac{x^2}{2}\cdot e^x + x\cdot e^x$, right?) would have $x\cdot e^x$ for its derivative.

Let's try it: $\left(\frac{x^2}{2}\cdot e^x + x\cdot e^x\right)' = x\cdot e^x + \frac{x^2}{2}\cdot e^x + 1\cdot e^x + x\cdot e^x$

$$= \frac{x^2}{2}e^x + 2xe^x + e^x$$

So since that's definitely not just $x\cdot e^x$, we can see that your rule didn't produce an antiderivative for $x\cdot e^x$.

9. Show that functions of the form $g(x) = \dfrac{x^n}{e^x}$ have local extremums where $x = n$. [Hint: You might warm up by working it with $n = 2$.]

First with $n=2$, $g(x) = \frac{x^2}{e^x}$. To find extrema, I'll take the derivative and set it equal to zero, then solve for x:

$$g'(x) = \frac{2x \cdot e^x - x^2 \cdot e^x}{(e^x)^2}$$

$$0 = \frac{2xe^x - x^2e^x}{e^{2x}}$$

$$0 = 2xe^x - x^2e^x$$

$$0 = x \cdot e^x(2-x)$$

So this happens when $x=0$ or $x=2$ (since e^x is never zero).

Now with n, $g(x) = \frac{x^n}{e^x}$, same drill:

$$g'(x) = \frac{nx^{n-1} \cdot e^x - x^n \cdot e^x}{(e^x)^2}$$

$$0 = \frac{n \cdot x^{n-1}e^x - x^ne^x}{e^{2x}}$$

$$0 = n \cdot x^{n-1} \cdot e^x - x^ne^x$$

$$0 = x^{n-1}e^x(n-x)$$

So this happens when $x=0$ or $x=n$, and it's clear that the sign of the derivative really does change when $x=n$, so it is indeed a local extreme.

10. Find a function of the form $f(x) = x^3 - b\,x^2 + c\,x - 2$ which has a local minimum where $x = 2$ and a point of inflection where $x = 1$.

$f'(x) = 3x^2 - 2bx + c$

$f''(x) = 6x - 2b$

To have a point of inflection where $x=1$, $f''(1)=0$, so

$0 = 6(1) - 2b$

$2b = 6$

$\underline{b = 3}$

And to have a minimum where $x = 2$, $f'(2)=0$, so

$0 = 3(2)^2 - 6(2) + c$

$0 = 12 - 12 + c$

$c = 0$

So $f(x) = x^3 - bx^2 + cx - 2$ should be

$f(x) = x^3 - 3x^2 - 2$