**Exam 3 Calc 1 7/20/2006**

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. Find a general antiderivative for the function $f(x) = \sqrt{x} + \frac{1}{x^2} + e^x$.

$f(x) = x^{1/2} + x^{-2} + e^x$

$$\boxed{F(x) = \frac{2}{3}x^{3/2} - x^{-1} + e^x + C}$$

Good

2. Find all critical number(s) of $f(x) = 4x^3 + x^4$.

Take deriv.: $f'(x) = 12x^2 + 4x^3$

set = zero: $0 = 4x^2(3+x)$

$x = 0$ or $x = -3$

3. Find the interval(s) on which $f(x) = 4x^3 + x^4$ is concave up.

$f''(x) = 24x + 12x^2$

$\emptyset = 24x + 12x^2$

$\emptyset = 12x(x+2)$

$\underline{X = 0}$ or $\underline{X = -2}$

Concave up $(-\infty, -2)$ and $(\emptyset, \infty)$

Concave down $(-2, \emptyset)$

Great

+ -2 - $\emptyset$ +

$f''(-3) = 36$ $f''(1) = 36$

$f''(-1) = -12$

4. If a company's cost to produce $x$ units of a particular good is given by $C(x) = 3000 + 5\sqrt{x}$, what are the company's marginal cost and average cost at a production level of 10000 units?

Marginal Cost: $C'(x) = 5 \cdot \frac{1}{2} x^{-1/2} = \frac{5}{2\sqrt{x}}$

$C'(10000) = \frac{5}{2\sqrt{10000}} = \frac{5}{200} = \frac{1}{40} = 0.025$

Average Cost: $c(x) = \frac{C(x)}{x} = \frac{3000 + 5\sqrt{x}}{x}$

$c(10000) = \frac{3000 + 5\sqrt{10000}}{10000} = \frac{3500}{10000} = \frac{7}{20} = .35$

5. Use Newton's Method with the initial approximation $x_1 = 0$ to find (to the nearest thousandth) $x_3$, the third approximation to a solution of the equation $e^x = 3x$.

$e^x = 3x$

$e^x - 3x = 0$

$f(x) = e^x - 3x$

$f'(x) = e^x - 3$

$x_2 = 0 - \frac{e^0 - 3(0)}{e^0 - 3}$

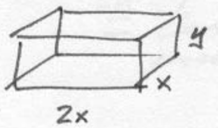
$= .5$

Great

$x_3 = .5 - \frac{e^{.5} - 3(.5)}{e^{.5} - 3}$

$\boxed{= .610}$

6. 1200 square inches of material are available to construct a box with a rectangular base twice as wide as it is long and an open top. What is the largest volume such a box can have?




Since 1200 $in^2$ is available for the surface,

$$1200 = \underbrace{(2x)(x)}_{\text{bottom}} + \underbrace{xy \cdot 2}_{\text{sides}} + \underbrace{2xy \cdot 2}_{\text{front + back}}$$

$$1200 = 2x^2 + 6xy$$

$$y = \frac{1200 - 2x^2}{6x} = \frac{200}{x} - \frac{x}{3}$$

So the volume of the box is:

$$V(x) = (2x)(x)\left(\frac{200}{x} - \frac{x}{3}\right) = 400x - \frac{2}{3}x^3$$

To maximize it, we'll take the derivative and set it equal to zero:

$$V'(x) = 400 - 2x^2$$

$$0 = 2(200 - x^2)$$

$$x = \pm\sqrt{200} \quad \text{(the positive root is realistic)}$$

The corresponding height is:

$$y = \frac{200}{\sqrt{200}} - \frac{\sqrt{200}}{3}$$

So the largest volume is:

$$V(\sqrt{200}) = (2\sqrt{200})(\sqrt{200})\left(\frac{200}{\sqrt{200}} - \frac{\sqrt{200}}{3}\right)$$

$$= 400\left(\frac{3\sqrt{200}}{3} - \frac{\sqrt{200}}{3}\right)$$

$$= \frac{800}{3}\sqrt{200}$$

Approaches $\infty$

7. Evaluate $\lim\limits_{x\to\infty} \dfrac{(\ln x)^2}{x}$.

Approachs $\infty$

$$\lim_{x\to\infty} \frac{(\ln x)^2}{x} \overset{L'H}{=} \lim_{x\to\infty} \frac{2(\ln x)\cdot\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{2\ln x}{x}$$

Well done.

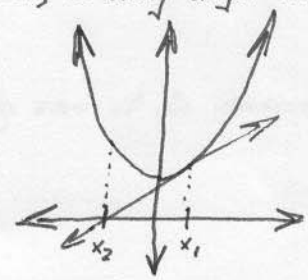
$$\lim_{x\to\infty} \frac{2\ln x}{x} \overset{L'H}{=} \lim_{x\to\infty} \frac{2\cdot\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{2}{x} = \boxed{0}$$

8. Bunny is a calculus student at Enormous State University, and she's wondering about something. Bunny says "So we learned about this Newton's method thingie in Calculus, and I pretty much get it, because it's mostly just a formula, right? But the professor said something about how we should know when it *doesn't* work. Isn't that unfair? If he teaches us something that doesn't work, shouldn't he get fired or something?"

Help Bunny out by **explaining** the circumstances under which Newton's Method fails to work, and why.

Actually, Bunny, it is important to know the limitations of Newton's Method if you're going to use it, so you can tell when something's going wrong!

The first sort of problem that can happen is with a function like $f(x) = x^2 + 1$. It never actually touches the axis, so any algorithm to find its roots will have problems. If you look at the graph at right, you can see that the second approximation, $x_2$, is even farther from being a root than $x_1$ was. If you keep going, your successive approximations will bounce around pretty randomly, without ever stabilizing anywhere.




Another issue is with functions like $f(x) = x^{1/3}$, so $f'(x) = \frac{1}{3}x^{-2/3}$. If you start with, say, $x_1 = 5$, then

$$x_2 = 5 - \frac{f(5)}{f'(5)} = -10$$

$$x_3 = -10 - \frac{f(-10)}{f'(-10)} = 20$$

And again the successive approximations keep bouncing farther and farther away from the root.

So Bunny, keep an eye out for that sort of weirdness when using Newton's Method, and be ready to look at a graph or use some other backup plan when it's not the right tool for the job!

9. Suppose a river's current has speed $u$ and a fish is swimming upstream with speed $v$ relative to the water. The energy $E$ expended in such a migration is

$$E(v) = \alpha \frac{v^3}{v-u}$$

where $\alpha$ is a positive constant. For what value of $v$ is $E$ minimized? [Borrowed from Blank & Kranz – and yes, migrating fish do in fact tend to swim at the predicted speed.]

To minimize, we'll take the derivative (with respect to v) and set equal to zero:

$$E'(v) = \alpha \cdot \frac{(3v^2)(v-u) - (v^3)(1)}{(v-u)^2}$$

$$0 = \alpha \cdot \frac{3v^3 - 3uv^2 - v^3}{(u-v)^2}$$

$$0 = \frac{2v^3 - 3uv^2}{(u-v)^2}$$ [It's only zero if the numerator's zero]

$$0 = v^2(2v - 3u)$$

$$v = 0 \quad \text{or} \quad 2v - 3u = 0$$

$$v = \frac{3}{2}u$$

So the fish should either give up or swim half again as fast as the current.

10. For which values of $a$ and $b$ will $\lim_{x\to\infty} \frac{(\ln x)^a}{x^b} = 0$ ?

Part I: Well, if a and b are both greater than zero, it basically works like #7, using L'Hôpital's Rule because it's an $\frac{\infty}{\infty}$ indeterminate form:

$$\lim_{x\to\infty} \frac{(\ln x)^a}{x^b} = \lim_{x\to\infty} \frac{a(\ln x)^{a-1}\cdot\frac{1}{x}}{b\,x^{b-1}} = \lim_{x\to\infty} \frac{a(\ln x)^{a-1}}{b\,x^b}$$

As long as $a-1$ is still positive, you can repeat this with similar results until eventually the numerator's exponent is zero or negative, which brings us to...

Part II: If $a \le 0$ and $b > 0$, we can write both terms in the denominator with positive exponents, so as x approaches infinity the quotient approaches 0.

Part III: On the other hand, if $b \le 0$, either L'Hôpital's Rule doesn't apply (when $a > 0$) or the reciprocal of Part I shows it heads for infinity.

Thus the limit will be 0 whenever $b > 0$, regardless of a.