Each problem is worth 10 points. For full credit indicate clearly how you reached your answer.

1. Which of the following are differential equations? Circle all that are.
   
   a) \( X = (A - \lambda I) \)
   
   b) \( y' = 3yt \)
   
   c) \( \frac{dy}{dt} + f(y)g(t) - 3 \cdot \frac{y}{30 + t} \)
   
   d) All men are mortal, and Socrates is a man, therefore all men are Socrates.
   
   e) \( y = Ae^{3t} - B \)

2. Sketch the phase line for the differential equation \( \frac{dP}{dt} = 0.3P \left( 1 - \frac{P}{2000} \right) \), and identify any equilibrium point(s) as sources, sinks, or nodes.

\[ 0.3P \left( 1 - \frac{P}{2000} \right) = 0 \]

- \( P = 0 \) is an equilibrium point.
- \( 1 - \frac{P}{2000} = 0 \) has no solution.
- \( P = 2000 \) is an equilibrium point.

The phase line shows:

- A source at \( P = 0 \) with a small upward slope.
- A sink at \( P = 2000 \) with a small downward slope.

Initial conditions and solution behavior should be noted in the question's context.
furnace is \[ \frac{dH}{dt} = k \left( 2000 - H \right) \], at least up until around where \( H(t) = 451° \). This equation has
general solutions of the form \( H(t) = 2000 - Ae^{-kt} \). If a book has a temperature of 70° when first
thrown into the furnace, and two minutes later has heated up to 200°, find a particular solution
(with values accurate to two significant figures) fitting this situation.

\[
t = 0 \quad T_{\text{book}} = 70 \quad \Rightarrow \quad H(0) = 2000 - A e^{0} = 2000 - A = 70 \\
\Rightarrow \quad A = 1930 \\
t = 2 \quad T_{\text{book}} = 200 \quad \Rightarrow \quad H(2) = 2000 - 1930 e^{-2k} = 200 \\
1800 = 1930 e^{-2k} \\
0.096 = e^{-2k} \Rightarrow k = 0.035
\]

\[ H(t) = 2000 - 1930 e^{-0.035t} \]

4. Jon’s water conditioner has been sabotaged so that it emits water with 5 grams of pink ink in
every gallon of water. If Jon is running water at a rate of 3 gallons per minute into his 10 gallon
sink, and the sink is half full of clean water before the ink solution begins to come out, write a
differential equation for the amount of pink ink in the sink \( t \) minutes after the ink starts to come
out, and an initial condition representing the situation.

\[
\frac{ds}{dt} = \frac{5 \text{ gal} \cdot 3}{5 \text{ gal}} - 3 \cdot \frac{s}{5 \text{ gal}}
\]

\[ \Rightarrow \quad \frac{ds}{dt} = 15 - \frac{3 \cdot s}{5} \]

with \( t = 0, \quad s = 0 \) as initial-value
5. Find a general solution to the differential equation \( \frac{dy}{dt} = \beta y - \alpha \).

\[
\int \frac{1}{\beta y - \alpha} \, dy = \int dt
\]

\[
\frac{1}{\beta} \ln |\beta y - \alpha| = t + C
\]

\[
\ln |\beta y - \alpha| = \beta t + C
\]

As sign absorbs the absolute value.

\[
\beta y - \alpha = Ae^{\beta t}
\]

\[
y = \frac{Ae^{\beta t} + \alpha}{\beta}
\]

6. Show that if \( f(x) \) is an integrable function, \( g(x) \) is a differentiable function, and \( u = g(x) \), then

\[
\int f(g(x)) \cdot g'(x) \, dx = \int f(u) \, du
\]

Proof: Well, let \( F(x) \) be a function for which \( F'(x) = f(x) \). By the chain rule, we get:

\[
[F(g(x))]' = F'(g(x)) \cdot g'(x)
\]

Antidifferentiating, we get:

\[
F(g(x)) + C = \int F'(g(x)) \cdot g'(x) \, dx
\]

And if we let \( g(x) = u \), then we get

\[
\int F'(g(x)) \cdot g'(x) \, dx = F(g(x)) + C
\]

\[
= F(u) + C
\]

\[
= \int f(u) \, du
\]

\[
= \int f(u) \, du. \quad \square
\]
7. Find a general solution to the differential equation \( \frac{dy}{dt} = e^{2t} + 5y \).

This is linear:

\[
\frac{dy}{dt} - 5y = e^{2t}
\]

\[
\int y \, dt - 5 \int y \, dt = \int e^{2t} \, dt - 5 \int e^{2t} \, dt
\]

\[
\int (y \cdot e^{-5t}) \, dt = \int e^{-3t} \, dt
\]

\[
y \cdot e^{-5t} = -\frac{1}{3} e^{-3t} + C
\]

\[
y = -\frac{1}{3} e^{-3t} + C \\
\]

\[
y(t) = -\frac{1}{3} e^{2t} + C \cdot e^{5t}
\]

8. Sketch the bifurcation diagram for the differential equation \( \frac{dy}{dt} = (y^2 - \alpha)(y^2 - 4) \). Include direction arrows on the phase lines and make clear the exact \( \alpha \) values where bifurcations occur.

Bifurcations when \( \alpha = 0 \), from two to three then four eq., and \( \alpha = 4 \), temporarily just two eq., and the arrows rearrange a bit.
9. Solve the initial value problem \( t \frac{dy}{dt} = y + \left( t^2 - y^2 \right)^{1/2} \), \( y(1) = 0 \) by using the substitution \( u = \frac{y}{t} \).

So substituting:

\[
t \left( \frac{u}{t} + t \frac{du}{dt} \right) = \left( ut \right) + \left( t^2 - (tu)^2 \right)^{1/2}
\]

\[
y + t^2 \frac{du}{dt} = ut + \sqrt{t^2 - t^2 u^2}
\]

\[
\frac{dt}{dt} = \frac{t}{t^2} \sqrt{1 - u^2}
\]

As long as \( t > 0 \)

Separable!

\[
\frac{1}{\sqrt{1-u^2}} \, du = \left( \frac{1}{t} \right) dt
\]

\[
\arcsin u = \ln t + C
\]

\[
u = \sin \left( \ln t + C \right)
\]

\[
\frac{y}{t} = \sin \left( \ln t + C \right)
\]

\[
y = t \cdot \sin \left( \ln t + C \right)
\]

General Solution.

\[
(0) = (1) \sin \left( \ln (1) + C \right)
\]

\[
0 = \sin C \quad \text{So } C = 0 \text{ works, as do multiples of } \pi.
\]

\[
y = t \cdot \sin \left( \ln t \right) \quad \text{Particular Solution.}
\]
10. Somebody tells you that if two functions are each solutions to a differential equation, then their product must also be a solution to that differential equation. Are they right? Why or why not?

*Nope. For a counterexample, notice that \( y' = y \)
has \( y_1 = e^t \) as a solution and \( y_2 = 2e^t \) as another solution, but \( y_3 = y_1 \cdot y_2 = e^t \cdot 2e^t = 2e^{2t} \) is not a solution.*