Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. Write the 3rd degree Maclaurin polynomial for $e^x$.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

2. Determine whether $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ converges or diverges.

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}} \text{ Which is a } p\text{-series with } p = 1 \text{ so } \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \text{ diverges}$$

Excellent!
3. Determine whether \( \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \) converges or diverges.

\[
\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \quad \text{Alternating Series Test}
\]

\[
\lim_{n \to \infty} \frac{1}{\sqrt{n}} = 0
\]

The derivative of \( \frac{1}{\sqrt{n}} \) is \( -\frac{1}{2} n^{-\frac{3}{2}} \). Because the derivative is negative, it is decreasing.

Therefore by the A.S.T., \( \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \) converges.

4. Determine whether the series \( \sum_{k=1}^{\infty} \frac{k^2 - 1}{k^3 + 4} \) converges or diverges.

Limit Comparison to \( \sum \frac{1}{k} \), the harmonic series, which diverges

\[
\lim_{k \to \infty} \frac{\frac{k^2 - 1}{k^3 + 4}}{\frac{1}{k}} = \lim_{k \to \infty} \frac{k^2 - 1}{k^3 + 4} \cdot \frac{k}{1} = \lim_{k \to \infty} \frac{k^3 - k}{k^3 + 4} = \lim_{k \to \infty} \frac{k^3}{k^3} \cdot \frac{1 - \frac{k^2}{k^3}}{1 + \frac{4}{k^3}} = 1
\]

So since that limit is finite and non-zero, both series do the same thing, and this one must also diverge.
5. Determine whether the series \( \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2} \) converges or diverges.

**Integral Test!**

\[
\int_{2}^{\infty} \frac{1}{x(\ln x)^2} \, dx = \lim_{b \to \infty} \int_{2}^{b} \frac{1}{x(\ln x)^2} \, dx = \lim_{b \to \infty} \left[ \frac{-1}{\ln x} \right]_{2}^{b} = \lim_{b \to \infty} \left( \frac{-1}{\ln b} - \frac{-1}{\ln 2} \right) = 0 + \frac{1}{\ln 2}
\]

So since that integral converges, the series also converges.

6. Determine the radius of convergence of the power series \( \sum \left( \frac{x}{3} \right)^k \).

**Ratio Test!**

\[
\lim_{k \to \infty} \left| \frac{\left( \frac{x}{3} \right)^{k+1}}{\left( \frac{x}{3} \right)^k} \right| = \lim_{k \to \infty} \left| \frac{x^{k+1}}{3^{k+1}} \cdot \frac{3^k}{x^k} \right| = \lim_{k \to \infty} \left| \frac{x}{3} \right|
\]

So the Ratio Test promises it converges provided \( \left| \frac{x}{3} \right| < 1 \) or \( \frac{|x|}{3} < 1 \) or \( |x| < 3 \), which means the radius of convergence is (3).
7. Biff is a calculus student at Enormous State University, and he's having some trouble. Biff says “Well, crap. I’m getting okay at finding these Taylor series and stuff, ‘cause I found there’s a formula in the book. But then there’s all these other things they bring in and I’m pretty lost. I might have to kill my roommate, ‘cause they say you get all A’s for a semester if your roommate dies. But if I can figure stuff out by the exam tomorrow, I guess I won’t have to do that. So like one of the things the prof said we needed to know was why the series \( x \) to the \( n \) sums up to 1 over 1 minus \( x \), and he said it was more an explaining thing about reasons than a bunch of calculating, but I’m not so good with reasons. Maybe I need to think more about the roommate option…”

Help Biff (and his roommate!) by explaining clearly how we can find the sum of \( \sum x^n \).

The geometric series test states that \( \sum_{n=1}^{\infty} ar^{n-1} \) converges to \( \frac{a}{1-r} \) iff \( |r|<1 \).

\( \sum_{n=0}^{\infty} x^n \) has an \( a \) of 1 and an \( r \) of \( x \), so

\( \sum_{n=0}^{\infty} x^n \) converges to \( \frac{1}{1-x} \) when \( |x|<1 \),

Which means the sum is \( \frac{1}{1-x} \) when \(-1<x<1\).

- Excellent
8. Use a Taylor series with at least 4 nonzero terms to approximate $\sqrt{e}$.

$$I \text{ know that } e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

$$\sqrt{e} = e^{1/2} \text{ so } x = \frac{1}{2}$$

that means $e^{1/2} \approx 1 + \frac{1}{2} + \frac{(\frac{1}{2})^2}{2} + \frac{(\frac{1}{2})^3}{6}$

so $\sqrt{e} \approx 1.646$

9. Use a Taylor series with at least 3 nonzero terms to approximate $\int_0^{0.2} \sin(x^2) \, dx$.

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

$$\sin(x^2) \approx x^2 - \frac{x^6}{6!} + \frac{x^{10}}{10!}$$

$$\int_0^{0.2} \sin(x^2) \, dx \approx \int_0^{0.2} x^2 - \frac{1}{6} x^6 + \frac{1}{120} x^{10} \, dx$$

$$\left[ \frac{1}{3} x^3 - \frac{1}{4!} x^7 + \frac{1}{13!} x^{11} \right]_0^{0.2}$$

$$\frac{1}{3} (0.2)^3 - \frac{1}{4!} (0.2)^7 + \frac{1}{13!} (0.2)^{11} \approx 0.00166193619$$

Well done.
10. Use a Taylor series to evaluate \( \lim_{x \to 0} \frac{x}{e^x - e^{-x}} \)

I know \( e^x \approx 1 + x \)
so \( e^{-x} \approx 1 - x \)

So \( \lim_{x \to 0} \frac{x}{e^x - e^{-x}} = \lim_{x \to 0} \frac{x}{1 + x - 1 + x} \)

\( \lim_{x \to 0} \frac{x}{2x} = \lim_{x \to 0} \frac{1}{2} = \frac{1}{2} \)

So \( \lim_{x \to 0} \frac{x}{e^x - e^{-x}} \approx \frac{1}{2} \)

Nice \( \frac{1}{2} \)