The basic steps in creating the model are:

1. **Identify the variables and parameters**:
   - Determine which variables are important and which are less significant.
   - Identify the parameters that affect the system.

2. **Formulate the assumptions**:
   - Define the assumptions that simplify the model.
   - Ensure that the assumptions are reasonable and support the model.

3. **Calculate the model parameters**:
   - Use data to fit the model parameters.
   - Adjust the parameters to improve model accuracy.

4. **Simulate the model**:
   - Test the model under various conditions.
   - Validate the model against real-world data.

5. **Interpret the results**:
   - Analyze the model outputs to draw conclusions.
   - Use the model to make predictions and decisions.

**What is a Model?**

A model is a simplified representation of a real-world system or process. It helps us understand complex systems and make predictions about their behavior. Models can be mathematical, physical, or conceptual.

**Modeling VIA Differential Equations**

This section covers how to use mathematics to study real-world phenomena. Differential equations are a powerful tool for modeling dynamic systems, allowing us to predict how variables change over time.

**Chapter 1: First-Order Differential Equations**

This chapter introduces the basics of first-order differential equations, which are used to model systems where the rate of change is proportional to the current state of the system.
Step 1 Clearly state the assumptions on which the model will be based. These assumptions should describe the relationships among the quantities to be studied.

Step 2 Completely describe the variables and parameters to be used in the model—"you can’t tell the players without a program."

Step 3 Use the assumptions formulated in Step 1 to derive equations relating the quantities in Step 2.

Step 1 is the "science" step. In Step 1, we describe how we think the physical system works or, at least, what the most important aspects of the system are. In some cases these assumptions are fairly speculative, as, for example, "rabbits don’t mind being overcrowded." In other cases the assumptions are quite precise and well accepted, such as "force is equal to the product of mass and acceleration." The quality of the assumptions determines the validity of the model and the situations to which the model is relevant. For example, some population models apply only to small populations in large environments, whereas others consider limited space and resources. Most important, we must avoid "hidden assumptions" that make the model seem mysterious or magical.

Step 2 is where we name the quantities to be studied and, if necessary, describe the units and scales involved. Leaving this step out is like deciding you will speak your own language without telling anyone what the words mean.

The quantities in our models fall into three basic categories: the independent variable, the dependent variables, and the parameters. In this book the independent variable is (almost) always time. Time is "independent" of any other quantity in the model. On the other hand, the dependent variables are quantities that are functions of the independent variable. For example, if we say that "position is a function of time," we mean that position is a variable that depends on time. We can vaguely state the goal of a model expressed in terms of a differential equation as "Describe the behavior of the dependent variable as the independent variable changes." For example, we may ask whether the dependent variable increases or decreases, or whether it oscillates or tends to a limit.

Parameters are quantities that don’t change with time (or with the independent variable) but that can be adjusted (by natural causes or by a scientist running the experiment). For example, if we are studying the motion of a rocket, the initial mass of the rocket is a parameter. If we are studying the amount of ozone in the upper atmosphere, then the rate of release of fluorocarbons from refrigerators is a parameter. Determining how the behavior of the dependent variables changes when we adjust the parameters can be the most important aspect of the study of a model.

In Step 3 we create the equations. Most of the models we consider are expressed as differential equations. In other words, we expect to find derivatives in our equations. Look for phrases such as "rate of change of ..." or "rate of increase of ..." since rate of change is synonymous with derivative. Of course, also watch for "velocity" (derivative of position) and "acceleration" (derivative of velocity) in models from physics. The word is means "equals" and indicates where the equality lies. The phrase "A is proportional to B" means $A = kB$, where $k$ is a proportionality constant (often a parameter in the model).
When we wanted the year 1790 (the year of the first census) is a convenient choice.

The year 1790 is also a convenient choice. The year 1790 corresponds to any
year and f(t) in millions of people. In this case we could let f(0) = 0, correspond to any

year, and f(t) in millions of people. In this case we could make a convenient choice.
The European population of the United States was 4,3 million people. If we make
an adjustment for the area of land covered by the land of the present day United States
and the area of land covered by the land of the present day United States, the

growth of our model can be measured in days or hours, but when we make

an adjustment for the area of land covered by the land of the present day United States,
The units for these quantities depend on the application. If we are modeling the
growth of population, then the units of the time "parameter" are years. If we are modeling
growth of population, then the units of the time "parameter" are years. If we are modeling
population growth, then the units of the time "parameter" are years. If we are modeling
population growth, then the units of the time "parameter" are years. If we are modeling

population growth, then the parameter t is often called the "growth-rate coefficient.

The parameter $\lambda$ is often called the "growth-rate coefficient."
Now let's express our assumption using this notation. The rate of growth of the population \( P \) is the derivative \( dP/dt \). Being proportional to the population is expressed as the product, \( kP \), of the population \( P \) and the proportionality constant \( k \). Hence our assumption is expressed as the differential equation

\[
\frac{dP}{dt} = kP.
\]

In other words, the rate of change of \( P \) is proportional to \( P \).

This equation is our first example of a differential equation. Associated with it are a number of adjectives that describe the type of differential equation that we are considering. In particular, it is a first-order equation because it contains only first derivatives of the dependent variable, and it is an ordinary differential equation because it does not contain partial derivatives. In this book we deal only with ordinary differential equations.

We have written this differential equation using the \( dP/dt \) Leibniz notation—the notation that we tend to use. However, there are many other ways to express the same differential equation. In particular, we could also write this equation as \( P' = kP \) or as \( \dot{P} = kP \). The “dot” notation is often used when the independent variable is time \( t \).

What does the model predict?

More important than the adjectives or how the equation is written is what the equation tells us about the situation being modeled. Since \( dP/dt = kP \) for some constant \( k \), \( dP/dt = 0 \) if \( P = 0 \). Thus the constant function \( P(t) = 0 \) is a solution of the differential equation. This special type of solution is called an equilibrium solution because it is constant forever. In terms of the population model, it corresponds to a species that is nonexistent.

If \( P(t_0) \neq 0 \) at some time \( t_0 \), then at time \( t = t_0 \)

\[
\frac{dP}{dt} = kP(t_0) \neq 0.
\]

As a consequence, the population is not constant. If \( k > 0 \) and \( P(t_0) > 0 \), we have

\[
\frac{dP}{dt} = kP(t_0) > 0,
\]

at time \( t = t_0 \) and the population is increasing (as one would expect). As \( t \) increases, \( P(t) \) becomes larger, so \( dP/dt \) becomes larger. In turn, \( P(t) \) increases even faster. That is, the rate of growth increases as the population increases. We therefore expect that the graph of the function \( P(t) \) might look like Figure 1.1.

The value of \( P(t) \) at \( t = 0 \) is called an initial condition. If we start with a different initial condition we get a different function \( P(t) \) as is indicated in Figure 1.2. If \( P(0) \) is negative (remembering \( k > 0 \)), we then have \( dP/dt < 0 \) for \( t = 0 \), so \( P(t) \) is initially decreasing. As \( t \) increases, \( P(t) \) becomes more negative. The picture below the \( t \)-axis is the flip of the picture above, although this isn’t “physically meaningful” because a negative population doesn’t make much sense.
is a function whose derivative is the product of f with (i)_d. Thus, we have:

\[ \frac{d^2f}{dp^2} = (i)_d \]

Now, consider a couple of ideas with various forms of the exponential function. Where that leads us, after a moment's reflection, is something that you will find in any text on the exponential and see for yourself. We can examine this expression by using the method of separation of variables. Essentially, we can transform it into an expression that is easy to guess. In our case, it is 

\[ f(t) = (i)_d \]

which has a derivative that is a product of f with (i)_d. Consequently, we can find a solution to the differential equation we must find a function

\[ \frac{df}{dt} = (i)_d \quad \text{for all} \]

\[ \frac{df}{dt} = \frac{iP}{dp} \]

which satisfies both conditions. That is, (i)_d

\[ (0)_d \quad \frac{df}{dt} = \frac{iP}{dp} \]

is called an initial-value problem. A solution to the initial-value problem is a function

\[ \frac{df}{dt} = (0)_d \quad \frac{df}{dt} = \frac{iP}{dp} \]

non of the above.

The partial differential equation of type (i)_d is called the partial differential equation. If we know the exact value of (0)_d and we want to predict the value of (0)_d, then we need more precise information about the function.

Analytic solutions of the differential equation

\[ \frac{df}{dt} = \frac{iP}{dp} \]

predicts population expressions. Then we can answer 'yes' as long as (0)_d < (0)_d. Each has a function that satisfies the differential equation of second degree. The graph of a function that satisfies

\[ \frac{df}{dt} = \frac{iP}{dp} \]

the differential equation

\[ \frac{df}{dt} = \frac{iP}{dp} \]

which satisfies and increases is called a graph.

\[ (0)_d \]

\[ (0)_d \]

Figure 1.1

Figure 1.2

CHAPTER 1 First-Order Differential Equations
there are other possible solutions, since \( P(t) = c e^{kt} \) (where \( c \) is a constant) yields
\[
d P/dt = c(k e^{kt}) = k(c e^{kt}) = kP(t).
\]
Thus \( d P/dt = kP \) for all \( t \) for any value of the constant \( c \).

We have infinitely many solutions to the differential equation, one for each value of \( c \). To determine which of these solutions is the correct one for the situation at hand, we use the given initial condition. We have
\[
P_0 = P(0) = c \cdot e^{k \cdot 0} = c \cdot e^0 = c \cdot 1 = c.
\]
Consequently, we should choose \( c = P_0 \), so a solution to the initial-value problem is
\[
P(t) = P_0 e^{kt}.
\]
We have obtained an actual formula for our solution, not just a qualitative picture of its graph.

The function \( P(t) \) is called the solution to the initial-value problem as well as a particular solution of the differential equation. The collection of functions \( P(t) = c e^{kt} \) is called the general solution of the differential equation because we can use it to find the particular solution corresponding to any initial-value problem. Figure 1.2 consists of the graphs of exponential functions of the form \( P(t) = c e^{kt} \) with various values of the constant \( c \), that is, with different initial values. In other words, it is a picture of the general solution to the differential equation.

The U.S. Population
As an example of how this model can be used, consider the U.S. census figures since 1790 given in Table 1.1.

<table>
<thead>
<tr>
<th>Year</th>
<th>( t )</th>
<th>Actual</th>
<th>( P(t) = 3.9 e^{0.03067t} )</th>
<th>Year</th>
<th>( t )</th>
<th>Actual</th>
<th>( P(t) = 3.9 e^{0.03067t} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1790</td>
<td>0</td>
<td>3.9</td>
<td>3.9</td>
<td>1930</td>
<td>140</td>
<td>122</td>
<td>286</td>
</tr>
<tr>
<td>1800</td>
<td>10</td>
<td>5.3</td>
<td>5.3</td>
<td>1940</td>
<td>150</td>
<td>131</td>
<td>388</td>
</tr>
<tr>
<td>1810</td>
<td>20</td>
<td>7.2</td>
<td>7.2</td>
<td>1950</td>
<td>160</td>
<td>151</td>
<td>528</td>
</tr>
<tr>
<td>1820</td>
<td>30</td>
<td>9.6</td>
<td>9.8</td>
<td>1960</td>
<td>170</td>
<td>179</td>
<td>717</td>
</tr>
<tr>
<td>1830</td>
<td>40</td>
<td>12</td>
<td>13</td>
<td>1970</td>
<td>180</td>
<td>203</td>
<td>975</td>
</tr>
<tr>
<td>1840</td>
<td>50</td>
<td>17</td>
<td>18</td>
<td>1980</td>
<td>190</td>
<td>226</td>
<td>1,320</td>
</tr>
<tr>
<td>1850</td>
<td>60</td>
<td>23</td>
<td>25</td>
<td>1990</td>
<td>200</td>
<td>249</td>
<td>1,800</td>
</tr>
<tr>
<td>1860</td>
<td>70</td>
<td>31</td>
<td>33</td>
<td>2000</td>
<td>210</td>
<td>281</td>
<td>2,450</td>
</tr>
<tr>
<td>1870</td>
<td>80</td>
<td>38</td>
<td>45</td>
<td>2010</td>
<td>220</td>
<td>-</td>
<td>3,320</td>
</tr>
<tr>
<td>1880</td>
<td>90</td>
<td>50</td>
<td>62</td>
<td>2020</td>
<td>230</td>
<td>-</td>
<td>4,520</td>
</tr>
<tr>
<td>1890</td>
<td>100</td>
<td>62</td>
<td>84</td>
<td>2030</td>
<td>240</td>
<td>-</td>
<td>6,140</td>
</tr>
<tr>
<td>1900</td>
<td>110</td>
<td>75</td>
<td>114</td>
<td>2040</td>
<td>250</td>
<td>-</td>
<td>8,340</td>
</tr>
<tr>
<td>1910</td>
<td>120</td>
<td>91</td>
<td>155</td>
<td>2050</td>
<td>260</td>
<td>-</td>
<td>11,300</td>
</tr>
<tr>
<td>1920</td>
<td>130</td>
<td>105</td>
<td>210</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
populations exist in a finite amount of space and with limited resources.

As we see from Figure 1.3, this model of $(i)\frac{dp}{dt} = 0.009673, 3.9^t$ population is given by

\[ \frac{6.3}{3.9} = 0.101 \]

Then we have

\[ 3.9^t = (10)^t \]

\[ 3.9^t \approx 3.9 \]

Let's see how well the unlimited growth model fits this data. We measure time in years and the population in millions of people. We also let $i = 0, 1, 2, \ldots$ represent the year 1790, 1800, and so on. The initial condition is $(i)\frac{dp}{dt} = 0.009673, 3, 9^t$. The correspondence initial-value problem is $\frac{dp}{dt} = 0.009673, 3.9^t, p(0) = 0$. If we set $\frac{dp}{dt} = 0.009673, 3.9^t, p(0) = 0$, then we have

\[ \frac{6.3}{3.9} = 0.101 \]

\[ 3.9^t = (10)^t \]

\[ 3.9^t \approx 3.9 \]

Thus our model predicts that the United States population is given by

\[ \frac{6.3}{3.9} = 0.101 \]

\[ 3.9^t = (10)^t \]

\[ 3.9^t \approx 3.9 \]

\[ \frac{6.3}{3.9} = 0.101 \]

\[ 3.9^t = (10)^t \]

\[ 3.9^t \approx 3.9 \]

\[ \frac{6.3}{3.9} = 0.101 \]

\[ 3.9^t = (10)^t \]

\[ 3.9^t \approx 3.9 \]

\[ \frac{6.3}{3.9} = 0.101 \]

\[ 3.9^t = (10)^t \]

\[ 3.9^t \approx 3.9 \]
Logistic Population Model

To adjust the exponential growth population model to account for a limited environment and limited resources, we add the assumptions:

- If the population is small, the rate of growth of the population is proportional to its size.
- If the population is too large to be supported by its environment and resources, the population will decrease. That is, the rate of growth is negative.

For this model, we again use

\[ t = \text{time (independent variable)}, \]
\[ P = \text{population (dependent variable)}, \]
\[ k = \text{growth-rate coefficient for small populations (parameter)}. \]

However, our assumption about limited resources introduces another quantity, the size of the population that corresponds to being "too large." This quantity is a second parameter, denoted by \( N \), that we call the "carrying capacity" of the environment. In terms of the carrying capacity, we are assuming that \( P(t) \) is increasing if \( P(t) < N \). However, if \( P(t) > N \), we assume that \( P(t) \) is decreasing.

Using this notation, we can restate our assumptions as:

- \( \frac{dP}{dt} \approx kP \) if \( P \) is small (first assumption).
- If \( P > N \), \( \frac{dP}{dt} < 0 \) (second assumption).

We also want the model to be "algebraically simple," or at least as simple as possible, so we try to modify the exponential model as little as possible. For instance, we might look for an expression of the form

\[ \frac{dP}{dt} = k \cdot (\text{something}) \cdot P. \]

We want the "something" factor to be close to 1 if \( P \) is small, but if \( P > N \) we want "something" to be negative. The simplest expression that has these properties is the function

\[ (\text{something}) = \left( 1 - \frac{P}{N} \right). \]

Note that this expression equals 1 if \( P = 0 \), and it is negative if \( P > N \). Thus our model is

\[ \frac{dP}{dt} = k \left( 1 - \frac{P}{N} \right) P. \]

This is called the logistic population model with growth rate \( k \) and carrying capacity \( N \). It is another first-order differential equation. This equation is said to be nonlinear because its right-hand side is not a linear function of \( P \) as it was in the exponential growth model.
The logistic equation of the population zero, and we again expect the population to level off at a
\( N \). Again, we can expect the population to level off at a
\( N \) when the population approaches the carrying capacity for the population to increase.

If we look at the graph of the function \( f(x) = \frac{1}{1 + e^{-x}} \) we see that it
Figure 1.2

\[ d \left( \frac{N}{d} - 1 \right) y = \frac{1}{d} \]

\[ 0 = d \]

\[ N = d \]

\[ d \]

The logistic equation of the population level off at a
\( N \) when the population approaches the carrying capacity for the population to increase.

If we look at the graph of the function \( f(x) = \frac{1}{1 + e^{-x}} \) we see that it
Figure 1.2

\[ d \left( \frac{N}{d} - 1 \right) y = \frac{1}{d} \]

\[ 0 = d \]

\[ N = d \]

\[ d \]
Finally, if \( P(0) < 0 \) (which does not make much sense in terms of populations), we also have \( dP/dt = f(P) < 0 \). Again we see that \( P(t) \) decreases, but this time it does not level off at any particular value since \( dP/dt \) becomes more and more negative as \( P(t) \) decreases.

Thus, just from a knowledge of the graph of \( f \), we can sketch a number of different solutions with different initial conditions, all on the same axes. The only information that we need is the fact that \( P = 0 \) and \( P = N \) are equilibrium solutions, \( P(t) \) increases if \( 0 < P < N \) and \( P(t) \) decreases if \( P > N \) or \( P < 0 \). Of course the exact values of \( P(t) \) at any given time \( t \) depend on the values of \( P(0) \), \( k \), and \( N \) (see Figure 1.7).

**Predator-Prey Systems**

No species lives in isolation, and the interactions among species give some of the most interesting models to study. We conclude this section by introducing a simple predator-prey system of differential equations where one species "eats" another. The most obvious difference between the model here and previous models is that we have two quantities that depend on time. Thus our model has two dependent variables that are both functions of time. Since both predator and prey begin with "p," we call the prey "rabbits," and the predators "foxes," and we denote the prey by \( R \) and the predators by \( F \). The assumptions for our model are:

- If no foxes are present, the rabbits reproduce at a rate proportional to their population, and they are not affected by overcrowding.
Considered together, this pair of equations is called a first-order system (only first order).

\[
\frac{dP}{dt} + \lambda P = \frac{rp}{P} \\
\frac{dP}{dt} - rP = \frac{gp}{P}
\]

Given these assumptions, we obtain the model:

This case: growth of rabbits, etc., so we add a term of the form \( \frac{rp}{P} \) from \( \frac{dP}{dt} + \lambda P = \frac{rp}{P} \). The growth assumption gives a similar term in the equation for \( N \). In this form the model is not linear, so it cannot be a solution of the linear equations of the previous section. (A simple term whose coefficients are functions of \( P \) and \( N \) doesn't allow the growth of rabbits and roses to interact, so we need a term that models the rate of interaction of the roses and rabbits. It's true that rabbits interact, but we need a term that models the rate at which the rabbits are eaten.)

The rate at which the rabbits are eaten is proportional to the rate at which the rabbits are eaten. Positive. When we formulate our model, we follow the convention that \( a, \lambda, g, \) and \( r \) are all positive. When we formulate our model, we follow the convention that \( a, \lambda, g, \) and \( r \) are all positive. When we formulate our model, we follow the convention that \( a, \lambda, g, \) and \( r \) are all positive.

Parameters are:
- \( P \) is our independent variable and our two dependent variables are \( P \) and \( N \).
- The rate at which the rabbits are eaten is proportional to the rate at which the rabbits are eaten. Positive.
- The rate at which the rabbits are eaten is proportional to the rate at which the rabbits are eaten. Positive.
- The rate at which the rabbits are eaten is proportional to the rate at which the rabbits are eaten. Positive.
- The rate at which the rabbits are eaten is proportional to the rate at which the rabbits are eaten. Positive.
- The rate at which the rabbits are eaten is proportional to the rate at which the rabbits are eaten. Positive.
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- The rate at which the rabbits are eaten is proportional to the rate at which the rabbits are eaten. Positive.
- The rate at which the rabbits are eaten is proportional to the rate at which the rabbits are eaten. Positive.
- The rate at which the rabbits are eaten is proportional to the rate at which the rabbits are eaten. Positive.
growth rate of the rabbit population. Also, since $\delta > 0$, the term $\delta RF$ is nonnegative. Consequently, an increase in the number of rabbits increases the growth rate of the fox population.

Although this model may seem relatively simple-minded, it has been the basis of some interesting ecological studies. In particular, Volterra and D’Ancona successfully used the model to explain the increase in the population of sharks in the Mediterranean during World War I when the fishing of “prey” species decreased. The model can also be used as the basis for studying the effects of pesticides on the populations of predator and prey insects.

A solution to this system of equations is, unlike our previous models, a pair of functions, $R(t)$ and $F(t)$, that describe the populations of rabbits and foxes as functions of time. Since the system is coupled, we cannot simply determine one of these functions first and then the other. Rather, we must solve both differential equations simultaneously. Unfortunately, for most values of the parameters, it is impossible to determine explicit formulas for $R(t)$ and $F(t)$. These functions cannot be expressed in terms of known functions such as polynomials, sines, cosines, exponentials, and the like. However, as we will see in Chapter 2, these solutions do exist, although we have no hope of ever finding them exactly. Since analytic methods for solving this system are destined to fail, we must use either qualitative or numerical methods to “find” $R(t)$ and $F(t)$.

The Analytic, Qualitative, and Numerical Approaches

Our discussion of the three population models in this section illustrates three different approaches to the study of the solutions of differential equations. The analytic approach searches for explicit formulas that describe the behavior of the solutions. Here we saw that exponential functions give us explicit solutions to the exponential growth model. Unfortunately, a large number of important equations cannot be handled with the analytic approach; there simply is no way to find an exact formula that describes the situation. We are therefore forced to turn to alternative methods.

One particularly powerful method of describing the behavior of solutions is the qualitative approach. This method involves using geometry to give an overview of the behavior of the model, just as we did with the logistic population growth model. We do not use this method to give precise values of the solution at specific times, but we are often able to use this method to determine the long-term behavior of the solutions. Frequently, this is just the kind of information we need.

The third approach to solving differential equations is numerical. The computer approximates the solution we seek. Although we did not illustrate any numerical techniques in this section, we will soon see that numerical approximation techniques are a powerful tool for giving us intuition regarding the solutions we desire.

All three of the methods we use have certain advantages, and all have drawbacks. Sometimes certain methods are useful while others are not. One of our main tasks as we study the solutions to differential equations will be to determine which method or combination of methods works in each specific case. In the next three sections, we elaborate on these three techniques.
Determine your solution to the actual data. Do you believe your prediction?

(a) Comparing the predicted area, and
determining the constant $k$.
(b) Solving the initial-value problem.

where in this case $A(t)$ is the land area occupied at time $t$. Make predictions about

$$A(t) = \frac{10}{A(t)}$$

of this land using an exponential growth model

mining land (including lands fore years from 1939-1974). Model the migration

The following table provides the land area in Australia colonized by the American

(i) For which values of $\lambda$ is $x$ decreasing?
(ii) For which values of $\lambda$ is $x$ increasing?
(iii) For which values of $\lambda$ is $x$ in equilibrium?

$$x = \frac{10^2}{x}$$

Consider the differential equation

(i) For which values of $d$ is the population decreasing?
(ii) For which values of $d$ is the population increasing?
(iii) For which values of $d$ is the population in equilibrium?

where $d$ is the population at time $t$.

$$d \left( 1 - \frac{10}{d} \right) \left( d^2 - 1 \right) = \frac{10^2}{d}$$

Consider the population model

(i) For which values of $d$ is the population decreasing?
(ii) For which values of $d$ is the population increasing?
(iii) For which values of $d$ is the population in equilibrium?

where $d$ is the population at time $t$.

$$\left( \frac{10^2}{d} - 1 \right) d^2 = \frac{10^2}{d}$$

EXERCISES FOR SECTION 1.1

CHAPTER 1 First-Order Differential Equations
<table>
<thead>
<tr>
<th>Year</th>
<th>Cumulative area occupied (km²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1939</td>
<td>32,800</td>
</tr>
<tr>
<td>1944</td>
<td>55,800</td>
</tr>
<tr>
<td>1949</td>
<td>73,600</td>
</tr>
<tr>
<td>1954</td>
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</tr>
<tr>
<td>1959</td>
<td>202,000</td>
</tr>
<tr>
<td>1964</td>
<td>257,000</td>
</tr>
<tr>
<td>1969</td>
<td>301,000</td>
</tr>
<tr>
<td>1974</td>
<td>584,000</td>
</tr>
</tbody>
</table>

(Note that there are many exponential growth models that you can form using this data. Is one a more reasonable model than the others? Note also that the area of Queensland is 1,728,000 km² and the area of Australia is 7,619,000 km².)

**Remark:** The American marine toad was introduced to Australia to control sugar cane beetles and, in the words of J. W. Hedgpath (see *Science, July 1993* and *The New York Times, July 6, 1993*),

Unfortunately the toads are nocturnal feeders and the beetles are abroad by day, while the toads sleep under rocks, boards and burrows. By night the toads flourish, reproduce phenomenally well and eat up everything they can find. The cane growers were warned by Walter W. Froggart, president of the New South Wales Naturalist Society, that the introduction was not a good idea and that the toads would eat the native ground fauna. He was immediately denounced as an ignorant meddlesome crank. He was also dead right.

Exercises 5–7 consider an elementary model of the learning process: Although human learning is an extremely complicated process, it is possible to build models of certain simple types of memorization. For example, consider a person presented with a list to be studied. The subject is given periodic quizzes to determine exactly how much of the list has been memorized. (The lists are usually things like nonsense syllables, randomly generated three-digit numbers, or entries from tables of integrals.) If we let \( L(t) \) be the fraction of the list learned at time \( t \), where \( L = 0 \) corresponds to knowing nothing and \( L = 1 \) corresponds to knowing the entire list, then we can form a simple model of this type of learning based on the assumption:

- The rate \( dL/dt \) is proportional to the fraction of the list left to be learned.

Since \( L = 1 \) corresponds to knowing the entire list, the model is

\[
\frac{dL}{dt} = k(1 - L),
\]

where \( k \) is the constant of proportionality.

---

Initial-value problem for the model in part (a):

\( y' = ky \) if the amount of the isotope present at time \( t = 0 \) is \( y_0 \), where the constant \( k \) is positive.

(6) Using this notation, write a model for the decay of a particular radioactive isotope.

\( y = y \) (radioactive substance), present at time \( t \) (dependent variable),

amount of particular radioactive isotope = \( y(t) \),

time (independent variable).

Note that the minus sign is used so that \( y \) is decreasing.

- \( y = \frac{y_0}{1 + k \cdot t} \) (decrease rate (rate law)).

8. Model radioactive decay using the notation.

On which radioactive isotope is used?

The amount of the isotope present, the proportionality constant depends only on the amount of the isotope and the amount of a quantity of a radioactive isotope decays in proportion to the amount of the substance which remained.

In Exercises 8-12, we consider the phenomenon of radioactive decay which, form ex-

\( \frac{dN}{dt} = -kN \)

\( \frac{dN}{dt} = -kN \)

where \( k \) and \( t \) are the fractions of the isotope left at time \( t \) by Jhillam and

\[
\begin{align*}
\frac{dN}{dt} \bigg|_{t=0} &= \frac{1}{T} \\
\frac{dN}{dt} \bigg|_{t=T} &= \frac{1}{T}
\end{align*}
\]

In Exercises 8-12, we consider the phenomenon of radioactive decay which, form ex-

(8) Which student has a faster rate of learning at time \( t \) if they both start memorizing?

(9) Which student has a faster rate of learning at time \( t \) if they both start memorizing?

(10) Which student has a faster rate of learning at time \( t \) if they both start memorizing?

(11) Which student has a faster rate of learning at time \( t \) if they both start memorizing?

(12) Which student has a faster rate of learning at time \( t \) if they both start memorizing?

(13) Which student has a faster rate of learning at time \( t \) if they both start memorizing?

(14) Which student has a faster rate of learning at time \( t \) if they both start memorizing?

(15) Which student has a faster rate of learning at time \( t \) if they both start memorizing?

(16) Which student has a faster rate of learning at time \( t \) if they both start memorizing?

(17) Which student has a faster rate of learning at time \( t \) if they both start memorizing?

(18) Which student has a faster rate of learning at time \( t \) if they both start memorizing?

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(23) Which student has a faster rate of learning at time \( t \) if they both start memorizing?

(24) Which student has a faster rate of learning at time \( t \) if they both start memorizing?

(25) Which student has a faster rate of learning at time \( t \) if they both start memorizing?

(26) Which student has a faster rate of learning at time \( t \) if they both start memorizing?

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(30) Which student has a faster rate of learning at time \( t \) if they both start memorizing?

(31) Which student has a faster rate of learning at time \( t \) if they both start memorizing?

(32) Which student has a faster rate of learning at time \( t \) if they both start memorizing?

(33) Which student has a faster rate of learning at time \( t \) if they both start memorizing?

(34) Which student has a faster rate of learning at time \( t \) if they both start memorizing?

(35) Which student has a faster rate of learning at time \( t \) if they both start memorizing?

(36) Which student has a faster rate of learning at time \( t \) if they both start memorizing?

(37) Which student has a faster rate of learning at time \( t \) if they both start memorizing?
9. The **half-life** of a radioactive isotope is the amount of time it takes for a quantity of radioactive material to decay to one-half of its original amount.

   (a) The half-life of Carbon 14 (C-14) is 5230 years. Determine the decay-rate parameter \( \lambda \) for C-14.

   (b) The half-life of Iodine 131 (I-131) is 8 days. Determine the decay-rate parameter for I-131.

   (c) What are the units of the decay-rate parameters in parts (a) and (b)?

   (d) To determine the half-life of an isotope, we could start with 1000 atoms of the isotope and measure the amount of time it takes 500 of them to decay, or we could start with 10,000 atoms of the isotope and measure the amount of time it takes 5000 of them to decay. Will we get the same answer? Why?

10. Carbon dating is a method of determining the time elapsed since the death of organic material. The assumptions implicit in carbon dating are that

   - Carbon 14 (C-14) makes up a constant proportion of the carbon that living matter ingests on a regular basis, and
   - once the matter dies, the C-14 present decays, but no new carbon is added to the matter.

Hence, by measuring the amount of C-14 still in the organic matter and comparing it to the amount of C-14 typically found in living matter, a "time since death" can be approximated. Using the decay-rate parameter you computed in Exercise 9, determine the time since death if

   (a) 88% of the original C-14 is still in the material.

   (b) 12% of the original C-14 is still in the material.

   (c) 2% of the original C-14 is still in the material.

   (d) 98% of the original C-14 is still in the material.

**Remark:** There has been speculation that the amount of C-14 available to living creatures has not been exactly constant over long periods (thousands of years). This makes accurate dates much trickier to determine.

11. In order to apply the carbon dating technique of Exercise 10, we must measure the amount of C-14 in a sample. Chemically, radioactive Carbon 14 (C-14) and regular carbon behave identically. How can we determine the amount of C-14 in a sample?  

   **[Hint: See Exercise 8.]**

12. The radioactive isotope I-131 is used in the treatment of hyperthyroid. When administered to a patient, I-131 accumulates in the thyroid gland, where it decays and kills part of that gland.

   (a) Suppose that it takes 72 hours to ship I-131 from the producer to the hospital. What percentage of the original amount shipped actually arrives at the hospital? (See Exercise 9.)
17. The following exercise concerns data for the population of lyme overs in Wytham Woods.

Discuss these assumptions. What is the differential equation that models the population of bread covered by mold? (Note that there is more than one reasonable model.)

In order to survive, mold must be in contact with the bread. When the fraction of bread covered by mold is large, the growth rate decreases. When the proportion covered is small, the fraction of the bread covered by mold increases at a rate proportional to the amount of bread covered by mold. (Note that there is more than one reasonable model.)

16. Consider the following assumptions concerning the fraction of bread covered by mold.

(3) Suppose that A is the area covered by mold. Then $A(t) = f(t)$, where $f(t)$ is a function of time.

(4) The growth rate of the mold population is proportional to the area of the bread covered by mold.

(5) Suppose that the growth rate of the mold population is proportional to the area of the bread covered by mold.

(6) Suppose that the growth rate of the mold population is proportional to the area of the bread covered by mold.

15. The following exercise is now extended. Suppose enough bread is present to support enough mold. How many breads can the population support?

14. Suppose that the growth rate parameters $r = 0.3$ and the carrying capacity $N = 2500$.

13. Suppose a species of fish in a particular lake has a population that is modeled by the logistic population model with growth rate $r$, carrying capacity $N$, and initial population $P(0) = 150$. Suppose $P(t) = 2500$.

12. The number of fish harvested each year is proportional to the square root of the number of fish in the lake. The number of fish harvested each year is proportional to the square root of the number of fish in the lake.

11. Suppose a species of fish in a particular lake has a population that is modeled by the logistic population model with growth rate $r$, carrying capacity $N$, and initial population $P(0) = 150$.

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3. The number of fish harvested each year is proportional to the square root of the number of fish in the lake. The number of fish harvested each year is proportional to the square root of the number of fish in the lake.

2. The number of fish harvested each year is proportional to the square root of the number of fish in the lake. The number of fish harvested each year is proportional to the square root of the number of fish in the lake.

1. The number of fish harvested each year is proportional to the square root of the number of fish in the lake. The number of fish harvested each year is proportional to the square root of the number of fish in the lake.
(a) What population model would you use to model this population?
(b) Can you approximate (or even make reasonable guesses for) the parameter values?
(c) What does your model predict for the population today?

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<th>Year</th>
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</thead>
<tbody>
<tr>
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<tr>
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<table>
<thead>
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<td>1958</td>
<td>62</td>
</tr>
<tr>
<td>1959</td>
<td>64</td>
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</table>

18. For the following predator-prey systems, identify which dependent variable, \( x \) or \( y \), is the prey population and which is the predator population. Is the growth of the prey limited by any factors other than the number of predators? Do the predators have sources of food other than the prey? (Assume that the parameters \( \alpha, \beta, \gamma, \delta, \) and \( N \) are all positive.)

(a) \[
\frac{dx}{dt} = -\alpha x + \beta xy \\
\frac{dy}{dt} = \gamma y - \delta xy
\]

(b) \[
\frac{dx}{dt} = \alpha x - \frac{x^2}{N} - \beta xy \\
\frac{dy}{dt} = \gamma y + \delta xy
\]

19. In the following predator-prey population models, \( x \) represents the prey, and \( y \) represents the predators.

(i) \[
\frac{dx}{dt} = 5x - 3xy \\
\frac{dy}{dt} = -2y + \frac{1}{2}xy
\]

(ii) \[
\frac{dx}{dt} = x - 8xy \\
\frac{dy}{dt} = -2y + 6xy
\]

(a) In which system does the prey reproduce more quickly when there are no predators (when \( y = 0 \)) and equal numbers of prey?

(b) In which system are the predators more successful at catching prey? In other words, if the number of predators and prey are equal for the two systems, in which system do the predators have a greater effect on the rate of change of the prey?

(c) Which system requires more prey for the predators to achieve a given growth rate (assuming identical numbers of predators in both cases)?
The standard form for a first-order differential equation is:

\[ \frac{dy}{dx} - \gamma y = \frac{1}{x} \]

or

\[ \frac{dx}{dy} + \gamma x = \frac{1}{y} \]

where \( \gamma \) is a constant. This equation is a function of the dependent variable and the independent variable. The left-hand side is the derivative of the dependent variable with respect to the independent variable. The right-hand side is a function of the independent variable.

What is a differential equation and what is a solution?

1.2 Analytic Techniques: Separation of Variables

\[ \frac{\gamma x^2}{x} - \frac{\gamma y}{y} = \frac{1}{x} \]


20. The system of differential equations is:

\[ \begin{align*}
\frac{dx}{dt} &= \gamma x \\
\frac{dy}{dt} &= \gamma y
\end{align*} \]