24. Suppose that a population can be accurately modeled by the logistic equation

$$\frac{dp}{dt} = 0.4p \left(1 - \frac{p}{30}\right).$$

(Note that the growth-rate parameter is 0.4 and the carrying capacity is 30.) Suppose that, at time $t = 5$, a disease is introduced into the population that kills 25% of the population per year. To adjust the model, we change the differential equation to

$$\frac{dp}{dt} = \begin{cases} 
0.4p \left(1 - \frac{p}{30}\right) & \text{for } 0 \leq t < 5; \\
0.4p \left(1 - \frac{p}{30}\right) - 0.25p & \text{for } t \geq 5.
\end{cases}$$

(a) Sketch the slope field for this equation using HPGeSolver.

(b) Using the slope field, sketch the graphs of a few representative solutions to this equation.

(c) Find formulas for the solutions of this equation for initial conditions $p(0) = 30$ and $p(0) = 20$.

(d) In a few sentences, describe the behavior of the solutions with initial conditions $p(0) = 30$ and $p(0) = 20$. (You can use either the sketches from the slope field or the formulas, but give a qualitative description of the solutions.)

1.4 NUMERICAL TECHNIQUE: EULER'S METHOD

The geometric concept of a slope field as discussed in the previous section is closely related to a fundamental numerical method for approximating solutions to a differential equation. Given an initial-value problem

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0,$$

we can get a rough idea of the graph of its solution by first sketching the slope field in the $ty$-plane and then, starting at the initial value $(t_0, y_0)$, sketching the solution by drawing a graph that is tangent to the slope field at each point along the graph. In this section we describe a numerical procedure that automates this idea. Using a computer or a calculator, we obtain numbers and graphs that approximate solutions to initial-value problems.

Numerical methods provide quantitative information about solutions even if we cannot find their formulas. There is also the advantage that most of the work can be done by machine. The disadvantage is that we obtain only approximations, not precise solutions. If we remain aware of this fact and are prudent, numerical methods become powerful tools for the study of differential equations. It is not uncommon to turn to numerical methods even when it is possible to find formulas for solutions. (Most of the graphs of solutions of differential equations in this text were drawn using numerical approximations even when formulas were available.)

The numerical technique that we discuss in this section is called Euler's method. A more detailed discussion of the accuracy of Euler's method as well as other numerical methods is given in Chapter 7.
Since we are given \( f(t, y) \), we can plot the slope field in the \( ty \)-plane. The idea of the method is to start at the point \((t_0, y_0)\) in the slope field and take steps dictated by the tangents in the slope field.

We begin by choosing a (small) step size \(\Delta t\). The slope of the approximate solution is updated every \(\Delta t\) units of \( t \). In other words, for each step, we move \(\Delta t\) units along the \(t\)-axis. The size of \(\Delta t\) determines the accuracy of the approximate solution.

Starting at \((t_0, y_0)\), our first step is to the point \((t_1, y_1)\) where \( t_1 = t_0 + \Delta t \) and our new point \((t_1, y_1)\) is the point on the line through \((t_0, y_0)\) with slope given by the slope field at \((t_0, y_0)\). The new point \((t_1, y_1)\) serves as an approximation to the solution at \( t_1 \).

Continuing, we use the slope field at the point \((t_1, y_1)\) to determine the next point \((t_2, y_2)\), and so on. We repeat the procedure, taking a step whose size is \(\Delta t\) units of \( t \) at each step. We reach the new point \((t_2, y_2)\) on the line segment that starts at \((t_1, y_1)\) and has slope \( f(t_1, y_1) \).

This method uses tangent line segments, given by the slope field, to approximate the graph of the solution. Consequently, at each stage we make a slight error (see Figure 1.30). Geometrically, we think of the method as producing a sequence of tiny line segments connecting \((t_0, y_0)\) to \((t_{n+1}, y_{n+1})\) (see Figure 1.30).

Basicall, we are stitching together little pieces of the slope field to form a graph that approximates our solution curve.

**Figure 1.30**

The graph of a solution obtained using Euler's method.
Euler's Method

To put Euler's method into practice, we need a formula for determining \((t_{k+1}, y_{k+1})\) from \((t_k, y_k)\). Finding \(t_{k+1}\) is easy. We specify the step size \(\Delta t\) at the outset, so

\[ t_{k+1} = t_k + \Delta t. \]

To obtain \(y_{k+1}\) from \((t_k, y_k)\), we use the differential equation. We know that the slope of the solution to the equation \(dy/dt = f(t, y)\) at the point \((t_k, y_k)\) is \(f(t_k, y_k)\), and Euler's method uses this slope to determine \(y_{k+1}\). In fact, the method determines the point \((t_{k+1}, y_{k+1})\) by assuming that it lies on the line through \((t_k, y_k)\) with slope \(f(t_k, y_k)\) (see Figure 1.31).

![Figure 1.31](image)

Now we can use our basic knowledge of slopes to determine \(y_{k+1}\). The formula for the slope of a line gives

\[ \frac{y_{k+1} - y_k}{t_{k+1} - t_k} = f(t_k, y_k). \]

Since \(t_{k+1} = t_k + \Delta t\), the denominator \(t_{k+1} - t_k\) is just \(\Delta t\), and therefore we have

\[ \frac{y_{k+1} - y_k}{\Delta t} = f(t_k, y_k) \]

\[ y_{k+1} = y_k + f(t_k, y_k) \Delta t. \]

This is the formula for Euler's method (see Figures 1.31 and 1.32).

![Figure 1.32](image)

Two successive steps of Euler's method.
Thus the first point \((1,1)\) on the graph of the approximate solution is \((0,1.1)\).

For the first step by \(y\)-coordinates, we have \(y_0 = 1\), \(y_1 = 1\), 
and \(y_1 = 1.1\). We compute the \(y\)-coordinate for the first step by

\[ \text{In this example, Euler's method is given by} \]

\[ y_1 = y_0 + \frac{\Delta y}{1 + \frac{\Delta y}{\Delta x}} = 1.1 \]

\[ \therefore y_1 = 1.1 \]

This equation is separable, and by separating and integrating we obtain the solution

\[ \int_{y_0}^{y_1} \frac{dy}{y} = \int_1^1 \frac{dx}{x} \]

\[ I = (0) \]

Consider the initial-value problem

\[ \frac{dy}{dx} = f(x, y) \]

\[ y(x_0) = y_0 \]

in which the differential equation in addition to defining how \(y\) is influenced by the independent variable \(x\), is able to inform some information we obtain in the known solution. Considered, we are able to gain some information whose solution we already know. In this way, we are able to compute the approximate solution of a differential equation.

To illustrate Euler's method, we first use it to approximate the solution to a differential equation.

Approximating an Autonomous Equation

1. Calculate the next point \((1+\theta, 1+\theta)\) using the formula \( \theta \).
2. Use the differential equation to compute the slope \(\frac{dy}{dx}(x, y)\).
3. From the previous point \((x, y)\) and the step size \(\theta\), compute the point

\[ \left( x + \theta, y + \frac{\theta \cdot f(x, y)}{\Delta y} \right) \]

Euler's method for \(\theta = \frac{1}{\Delta y}\)
To compute the $y$-coordinate $y_2$ for the second step, we now use $y_1$ rather than $y_0$. That is,

$$y_2 = y_1 + (2y_1 - 1)\Delta t = 1.1 + (1.2)0.1 = 1.22,$$

and the second point for our approximate solution is $(t_2, y_2) = (0.2, 1.22)$.

Continuing this procedure, we obtain the results given in Table 1.3. After ten steps, we obtain the approximation of $y(1)$ by $y_{10} = 3.596$. (Different machines use different algorithms for rounding numbers, so you may get slightly different results on your computer or calculator. Keep this fact in mind whenever you compare the numerical results presented in this book with the results of your calculation.) Since we know that

$$y(1) = \frac{e^2 + 1}{2} \approx 4.195,$$

the approximation $y_{10}$ is off by slightly less than 0.6. This is not a very good approximation, but we’ll soon see how to avoid this (usually). The reason for the error can be seen by looking at the graph of the solution and its approximation. The slope field for this differential equation always lies below the graph (see Figure 1.33), so we expect our approximation to come up short.

Using a smaller step size usually reduces the error, but more computations must be done to approximate the solution over the same interval. For example, if we halve the step size in this example ($\Delta t = 0.05$), then we must calculate twice as many steps, since $t_1 = 0.05$, $t_2 = 0.1, \ldots, t_{20} = 1.0$. Again we start with $(t_0, y_0) = (0, 1)$ as specified by the initial condition. However, with $\Delta t = 0.05$, we obtain

$$y_1 = y_0 + (2y_0 - 1)\Delta t = 1 + (1)0.05 = 1.05.$$

This step yields the point $(t_1, y_1) = (0.05, 1.05)$ on the graph of our approximate

<table>
<thead>
<tr>
<th>$k$</th>
<th>$t_k$</th>
<th>$y_k$</th>
<th>$f(t_k, y_k)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0.1</td>
<td>1.100</td>
<td>1.20</td>
</tr>
<tr>
<td>2</td>
<td>0.2</td>
<td>1.220</td>
<td>1.44</td>
</tr>
<tr>
<td>3</td>
<td>0.3</td>
<td>1.364</td>
<td>1.73</td>
</tr>
<tr>
<td>4</td>
<td>0.4</td>
<td>1.537</td>
<td>2.07</td>
</tr>
<tr>
<td>5</td>
<td>0.5</td>
<td>1.744</td>
<td>2.49</td>
</tr>
<tr>
<td>6</td>
<td>0.6</td>
<td>1.993</td>
<td>2.98</td>
</tr>
<tr>
<td>7</td>
<td>0.7</td>
<td>2.292</td>
<td>3.58</td>
</tr>
<tr>
<td>8</td>
<td>0.8</td>
<td>2.650</td>
<td>4.30</td>
</tr>
<tr>
<td>9</td>
<td>0.9</td>
<td>3.080</td>
<td>5.16</td>
</tr>
<tr>
<td>10</td>
<td>1.0</td>
<td>3.596</td>
<td></td>
</tr>
</tbody>
</table>
There are always decisions to be made such as the choice of the step size $h$. However, in the end, we obtain a much better approximation to the solution (see Table 1.2).


$\text{Table 1.2}$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>$y'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>0.1</td>
<td>0.5001</td>
<td>0.1</td>
</tr>
<tr>
<td>0.2</td>
<td>0.5501</td>
<td>0.05</td>
</tr>
<tr>
<td>0.3</td>
<td>0.6501</td>
<td>0.03</td>
</tr>
<tr>
<td>0.5</td>
<td>0.8501</td>
<td>0.02</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

If we carefully compare the final results of our two computations, we see that $x = 1$.

Euler's method (to three decimal places) for $y' = x$ with $y(0) = 0.5$ is $y = x^2$, whereas the exact solution is $y = \frac{x^3}{3}$. The error in the first approximation is slightly less than 0.6, whereas the error in the second approximation is 0.33.

$E_{\text{approx}}(x) = 3.844$. The error in the first approximation is slightly less than 0.6, whereas the error in the second approximation is 0.33.

Accordingly, we believe the error by halving the step size. This type of improvement is typical of numerical methods. This is due to the fact that the second approximation is $0.33t$. In Chapter 1, we approximate the integral $\int_0^1 f(x) \, dx$ with $\frac{1}{n}$, $n = 1, 2, 3, \ldots$, $f(x)$. If we carefully compute the final results of our two computations, we see that $x = 1$.

**Table 1.4**

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>$y'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>0.1</td>
<td>0.5001</td>
<td>0.1</td>
</tr>
<tr>
<td>0.2</td>
<td>0.5501</td>
<td>0.05</td>
</tr>
<tr>
<td>0.3</td>
<td>0.6501</td>
<td>0.03</td>
</tr>
<tr>
<td>0.5</td>
<td>0.8501</td>
<td>0.02</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

For $x = 1$, the value of $y$ is $1$. The value of $y'$ is $1$. Therefore, the value of $y$ is $1.05$. This is the type of calculation used in Chapter 1. Now we have the point $(1.05, 1.05)$.

To find the next value of $y$, we compute

$y(1.05) + y'(1.05) \Delta x = y(1.10) = 1.05(1 - 1.10) + 1.10 = 1.20$.

By Euler's method with $y(0) = 0.0$, we find that the approximation produced

$y(1.10) = 1.20$.

The graph of the solution is shown in Figure 1.3.
Table 1.5  
Euler’s method (to four decimal places) for \( dy/dt = 2y - 1 \), \( y(0) = 1 \) with \( \Delta t = 0.01 \)

<table>
<thead>
<tr>
<th>( k )</th>
<th>( t_k )</th>
<th>( y_k )</th>
<th>( f(t_k, y_k) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0.01</td>
<td>1.0100</td>
<td>1.0200</td>
</tr>
<tr>
<td>2</td>
<td>0.02</td>
<td>1.0202</td>
<td>1.0404</td>
</tr>
<tr>
<td>3</td>
<td>0.03</td>
<td>1.0306</td>
<td>1.0612</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
</tr>
<tr>
<td>98</td>
<td>0.98</td>
<td>3.9817</td>
<td>6.9633</td>
</tr>
<tr>
<td>99</td>
<td>0.99</td>
<td>4.0513</td>
<td>7.1026</td>
</tr>
<tr>
<td>100</td>
<td>1.00</td>
<td>4.1223</td>
<td>\</td>
</tr>
</tbody>
</table>

A Nonautonomous Example

Note that it is the value \( f(t_k, y_k) \) of the right-hand side of the differential equation at \((t_k, y_k)\) that determines the next point \((t_{k+1}, y_{k+1})\). The last example was an autonomous differential equation, so the right-hand side \( f(t_k, y_k) \) depended only on \( y_k \). However, if the differential equation is nonautonomous, the value of \( t_k \) also plays a role in the computations.

To illustrate Euler’s method applied to a nonautonomous equation, we consider the initial-value problem

\[
\frac{dy}{dt} = -2ty^2, \quad y(0) = 1.
\]

This differential equation is also separable, and we can separate variables to obtain the solution

\[
y(t) = \frac{1}{1 + t^2}.
\]

We use Euler’s method to approximate this solution over the interval \( 0 \leq t \leq 2 \). The value of the solution at \( t = 2 \) is \( y(2) = 1/5 \). Again, it is interesting to see how close we come to this value with various choices of \( \Delta t \). The formula for Euler’s method is

\[
y_{k+1} = y_k + f(t_k, y_k) \Delta t = y_k - (2t_k y_k^2) \Delta t
\]

with \( t_0 = 0 \) and \( y_0 = 1 \). We begin by approximating the solution from \( t = 0 \) to \( t = 2 \) using just four steps. This involves so few computations that we can perform the arithmetic “by hand.” To cover an interval of length 2 in four steps, we must use \( \Delta t = 2/4 = 1/2 \). The entire calculation is displayed in Table 1.6. Note that we end up

Table 1.6  
Euler’s method for \( dy/dt = -2ty^2 \), \( y(0) = 1 \) with \( \Delta t = 1/2 \)

<table>
<thead>
<tr>
<th>( k )</th>
<th>( t_k )</th>
<th>( y_k )</th>
<th>( f(t_k, y_k) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1/2</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1/2</td>
<td>-1/2</td>
</tr>
<tr>
<td>3</td>
<td>3/2</td>
<td>1/4</td>
<td>-3/16</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>5/32</td>
<td></td>
</tr>
</tbody>
</table>
Table 1.8

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>( \Delta y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>2</td>
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</tbody>
</table>

Table 1.7

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>( \Delta y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
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<td>0</td>
</tr>
<tr>
<td>0.1</td>
<td>1</td>
<td>1</td>
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<tr>
<td>0</td>
<td>2</td>
<td>2</td>
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</table>

1000 = \( x \) with \( 2^3 = \frac{x}{2} \)

Table I.8

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>( \Delta y )</th>
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<tbody>
<tr>
<td>0.2</td>
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<td>0</td>
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Table I.7

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<tr>
<th>x</th>
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<td>0.2</td>
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<td>0</td>
</tr>
<tr>
<td>0.1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

Note the converse of the approximation to the actual value is slow. We prove \( o(2000) = 0.199937 \) (see Tables I.7 and I.8).

The graph of the solution to the initial-value problem is shown in Figure I.3.4.

As before, choosing smaller values of \( h \) yields better approximations. For example:

\[ i = 1 \]

The error method produced by Euler's method is used to find the initial-value of the solution graphed in Figure I.3.4.

**CHAPTER 1** First Order Differential Equations
An RC Circuit with Periodic Input

Recall from Section 1.3 that the voltage $v_c$ across the capacitor in the simple circuit shown in Figure 1.35 is given by the differential equation

$$\frac{d v_c}{d t} = \frac{V(t) - v_c}{RC}$$

where $R$ is the resistance, $C$ is the capacitance, and $V(t)$ is the source or input voltage. We have seen how we can use slope fields to give a qualitative sketch of solutions. Using Euler's method we can also obtain numerical approximations of the solutions.

Suppose we consider a circuit where $R = 0.5$ and $C = 1$. (The usual units are "ohms" for resistance and "farads" for capacitance. We choose these numbers so that the numbers in the solution work out nicely. A 1 farad capacitor would be extremely large.) Then the differential equation is

$$\frac{d v_c}{d t} = \frac{V(t) - v_c}{0.5} = 2(V(t) - v_c).$$

To understand how the voltage $v_c$ varies if the voltage source $V(t)$ is periodic in time, we consider the case where $V(t) = \sin(2\pi t)$. Consequently, the voltage oscillates between $-1$ and $1$ once each unit of time (see Figure 1.36). The differential equation is now

$$\frac{d v_c}{d t} = -2v_c + 2 \sin(2\pi t).$$

From the slope field for this equation (see Figure 1.37), we might predict that the solutions oscillate. Using Euler's method applied to this equation for several different initial conditions, we see that the solutions do indeed oscillate. In addition, we see that they also approach each other and collect around a single solution (see Figure 1.38). This uniformity of long-term behavior is not so easily predicted from the slope field alone.

$$\frac{d v_c}{d t} = -2v_c + 2 \sin(2\pi t).$$

Figure 1.37
Slope field for $d v_c/d t = -2v_c + 2 \sin(2\pi t)$.

Figure 1.38
Graphs of approximate solutions to $d v_c/d t = -2v_c + 2 \sin(2\pi t)$ obtained using Euler's method.
We have now introduced examples of all three of the fundamental methods for solving differential equations. The Big Three:

- Euler's method
- Improved Euler's method
- Runge-Kutta method

In the next section we present theoretical results that help identify when numerical approximations have gone awry.

In general, numerical methods work beautifully. But they sometimes fail. The problem is typical of the use of numerical methods in the study of differential equations.

Let's look at the general solution.

If it fails, the solution becomes very large. In most cases, the right-hand side becomes very large as the increase, and consequently, the solution diverges. This difficulty arises in Euler's method for this equation because of the term $\rho$ on the right-hand side.

In the limit, the solution diverges. This is typically how numerical methods work.

Consider the differential equation

$$x' = \frac{1}{x-1}$$

We can see that solutions can accumulate if $x$ is close to 1. In such cases, the error grows without bound.

It is very rare that any numerical approximation scheme is exact. For instance,
Stephen Smale (1930– ) is one of the founders of modern-day dynamical systems theory. In the mid-1960s, Smale began to advocate taking a more qualitative approach to the study of differential equations, as we do in this book. Using this approach, he was among the first mathematicians to encounter and analyze a "chaotic" dynamical system. Since this discovery, scientists have found that many important physical systems exhibit chaos.

Smale's research has spanned many disciplines, including economics, theoretical computer science, mathematical biology, as well as many subareas of mathematics. In 1966 he was awarded the Fields Medal, the equivalent of the Nobel Prize in mathematics. He is currently Professor Emeritus at the University of California, Berkeley.

what we want to know about the solutions. Often all three methods "work," but a great deal of labor can be saved if we think first about which method gives the most direct route to the information we need.

EXERCISES FOR SECTION 1.4

In Exercises 1–4, use Euler's Method to perform Euler's method with the given step size $\Delta t$ on the given initial-value problem over the time interval specified. Your answer should include a table of the approximate values of the dependent variable. It should also include a sketch of the graph of the approximate solution.

1. $\frac{dy}{dt} = 2y + 1, \quad y(0) = 3, \quad 0 \leq t \leq 2, \quad \Delta t = 0.5$
2. $\frac{dy}{dt} = t - y^2, \quad y(0) = 1, \quad 0 \leq t \leq 1, \quad \Delta t = 0.25$
3. $\frac{dy}{dt} = y^2 - 2y + 1, \quad y(0) = 2, \quad 0 \leq t \leq 2, \quad \Delta t = 0.5$
4. $\frac{dy}{dt} = \sin y, \quad y(0) = 1, \quad 0 \leq t \leq 3, \quad \Delta t = 0.5$

In Exercises 5–8, use Euler's method with the given step size $\Delta t$ to approximate the solution to the given initial-value problem over the time interval specified. Your answer should include a table of the approximate values of the dependent variable. It should also include a sketch of the graph of the approximate solution.

5. $\frac{dw}{dt} = (3 - w)(w + 1), \quad w(0) = 4, \quad 0 \leq t \leq 5, \quad \Delta t = 1.0$
6. $\frac{dw}{dt} = (3 - w)(w + 1), \quad w(0) = 0, \quad 0 \leq t \leq 5, \quad \Delta t = 0.5$
Consider the polynomial $d = \frac{1}{p} \frac{\partial}{\partial p}$, using appropriate technology.

Initial conditions over the interval $[0, \infty)$ and $C = 0$.

Suppose $z$ is oscillating periodically. If $f = 4$, then $\cos z = (i) f(t) = (i) f(t) \cos z$.

In Exercises 14-17, we consider the RC circuit equation.

How do the graphs of the approximate solutions relate to the graphs of the actual solution? What predictions do you make about the actual solution to the initial-value problems? If there are solutions, graph all three solutions.

Using Euler's method, compare the different approximate solutions corresponding to $i = 1$.

Consider the initial-value problem.

What predictions do you make about the actual solution to the initial-value problem? If there are solutions, graph all three solutions.

Using Euler's method, compare the different approximate solutions corresponding to $i = 1$.

The approximate solution given by Euler's method and compare your conclusions with your results in Exercises 6. What's wrong with this case? How would you avoid the difficulties that arise in this case?

Compare your answers to Exercises 5 and 6 and explain your observations.

$z = \frac{1}{p} \frac{\partial}{\partial p}$