There are many possible ways to choose L_1 and L_2, one of which is the following.

**Theorem:** There is a path L_2 of path connected open subsets of S', whose union is S', and for which p: S' --+ F, maps each path component of p(', S') isomorphic.

**Lemma:** If there is a path L_1 of path connected open subsets of S', whose union is S', and for which p: S' --+ F, maps each path component of p(', S') isomorphic.

The primary properties of the covering projection p: S' --+ F, are needed to produce a lifting of a homotopy f: I --+ F, and a homotopy g: I --+ S' to a covering homotopy F: S' --+ I x I.

We shall be particularly interested in lifting a path of the real line R to the real line R.

For which paths f: X --+ S', is called a covering function of f: X --+ S', a continuous function f: X --+ S', a space and f: X --+ S', a covering projection of f: X --+ S', will be instrumental in the study of the additive group Z of integers. The function p: S' --+ F, defined by p(x) = (cos(2πx), sin(2πx)), exhibits this section shows that the fundamental group of a circle is isomorphic to Z.

**The Fundamental Group of S'**

(a) Prove that each weakly contractible space is simply connected.

(b) Give an example of a weakly contractible space that is not contractible.

(c) In the case that a weak contraction is not required to leave the base point x fixed.

Thus the difference between a contraction on X and a weak contraction on X is

The function G is called a weak contraction.
Then $U_1$ and $U_2$ are clearly path connected open subsets of $S^1$ whose union equals $S^1$. From the definition of the covering projection $p$, it follows easily that

$$p^{-1}(U_1) = \bigcup_{k=-\infty}^{\infty} (k - 1/2, k + 1/4),$$

$$p^{-1}(U_2) = \bigcup_{k=-\infty}^{\infty} (k, k + 3/4).$$

Note that the path components of $p^{-1}(U_j)$ are the intervals $(k - 1/2, k + 1/4)$, $k$ an integer, each of which is mapped by $p$ homeomorphically onto $U_j$. Similarly, $p$ maps the path components $(k, k + 3/4)$ of $p^{-1}(U_2)$ homeomorphically onto $U_2$. This completes the proof of the lemma.

**Theorem 9.6: The Covering Path Property**  If $\alpha: I \to S^1$ is a path with initial point $I$, then there is a lifting of $\alpha$ to a unique covering path $\tilde{\alpha}: I \to \mathbb{R}$ with initial point $0$.

**Proof:** The proof rests on the following intuitive idea. Subdivide the range of the path $\alpha$ into connected sections so that each section is contained either in $U_1$ or $U_2$, the sets prescribed in the proof of the lemma. If a certain section is contained in $U_1$, we choose one of the intervals $A = (k - 1/2, k + 1/4)$ and consider the restriction $p|_A$ of $p$ to this interval. Since this restriction is a homeomorphism, we can compose the inverse of $p|_A$ with the given section of $\alpha$ to "lift" this section to a section of a path in $\mathbb{R}$. Sections lying in $U_2$ are lifted similarly. Being careful to have the terminal point of one lifted section agree with the initial point of the next lifted section will insure a continuous lifting of the entire path.
\[ \begin{align*}
\text{Let } B = I - I \text{ with } I = \{ \lambda \in \mathbb{R}^+ \mid \lambda \geq 0 \}, \\
\text{and } \gamma \text{ be a triangle in } I \text{ with } \gamma \cap B = \emptyset.
\end{align*} \]

\[ \phi : [\gamma] \rightarrow [\gamma]. \]

\[ [\gamma] \times [\gamma] \rightarrow [\gamma]. \]

\[ \phi : [\gamma] \rightarrow [\gamma]. \]

\[ \text{Proof:} \]

\[ \text{The Covariant Homotopy Property.} \]

\[ \text{Theorem 9.7:} \]
where
\[0 = t_0 < t_1 < \cdots < t_n = 1, \quad 0 = s_0 < s_1 < \cdots < s_m = 1\]
so that \( H \) maps any of the prescribed rectangles into either \( U_1 \) or \( U_2 \). Since \( H(0, 0) = 1 \) is not in \( U_2 \), then \( H \) must map the first rectangle \([t_0, t_1] \times [s_0, s_1]\) into \( U_1 \). Letting \( A_j = (-1/2, 1/4) \) as before, define \( \tilde{H} \) on \([t_0, t_1] \times [s_0, s_1]\) by
\[
\tilde{H}(t, s) = (p \mid A_j)^{-1} H(t, s).
\]

The definition of \( \tilde{H} \) is extended over the rectangles \([t_k, t_{k+1}] \times [s_0, s_1]\) as in the proof of the Covering Path Property, being sure that the definitions agree on the edges between rectangles. This defines \( \tilde{H} \) on the strip \([0, 1] \times [s_0, s_1]\). Next, \( H \) is defined in an analogous way on the strip \([0, 1] \times [s_1, s_2]\), with the definitions agreeing on the edges between rectangles. This argument extends inductively in a straightforward way to complete the proof.

**Definition:** For a loop \( \alpha \) in \( S^1 \) with base point 1, the Covering Path Property specifies a unique covering path \( \tilde{\alpha} \) of \( \alpha \) with \( \tilde{\alpha}(0) \) equal to 0. Since
\[(\cos 2\pi \tilde{\alpha}(1), \sin 2\pi \tilde{\alpha}(1)) = p\tilde{\alpha}(1) = \alpha(1) = 1\]
it follows that \( \tilde{\alpha}(1) \) must be an integer. This integer is called the **degree** of the loop \( \alpha \) and is denoted \( \text{deg}(\alpha) \).

Intuitively, one thinks of the degree of a loop \( \alpha \) as the net number of times that \( \alpha \) “wraps” the interval \([0, 1]\) around \( S^1 \). Counterclockwise wrappings are counted as positive and clockwise ones as negative. The next theorem shows that the degree of a loop completely determines its equivalence class in \( \pi_1(S^1, 1) \).

**Theorem 9.8:** For loops \( \alpha, \beta \) in \( S^1 \) with base point 1, \([\alpha] = [\beta]\) if and only if \( \text{deg}(\alpha) = \text{deg}(\beta) \).

**Proof:** Suppose first that \([\alpha] = [\beta]\) so that \( \alpha \) and \( \beta \) are equivalent loops in \( S^1 \). Let \( F : I \times I \rightarrow S^1 \) be a homotopy demonstrating the equivalence of \( \alpha \) and \( \beta \):
\[
F(\ast, 0) = \alpha, \quad F(\ast, 1) = \beta, \quad F(0, s) = F(1, s) = 1, \quad s \in I.
\]
The Covering Homotopy Property insures the existence of a unique covering homotopy \( \tilde{F} \) of \( F \) such that \( \tilde{F}(0, 0) = 0 \). For \( s \) in \( I \),
\[
p\tilde{F}(0, s) = F(0, s) = 1,
\]
and thus has degree $p^n$. Thus $d = 1$. Let $\gamma = (1)^n$. Thus $d = 1$.

has a covering path if the function

$I \in 1 \subseteq \cdots \subseteq [0, 1]$,

is a covering path of the function

$d: (I, S) \sqcup [0, 1] \ni \gamma \mapsto \delta \gamma = ((1)^n)$

degree of a loop equivalence class $[\gamma]$ where $\gamma$ assigns the integer

Theorem 9.8 shows that $d$ is well-defined and one-to-one. To see that it is surjective,

Proof: Consider the degree function $d: (I, S) \sqcup [0, 1] \ni \gamma \mapsto \delta \gamma = ((1)^n)$

of integers.

The fundamental group $\pi_1(S, 1)$ is isomorphic to the additive group $\mathbb{Z}$.

Theorem 9.9: The fundamental group $\pi_1(S, 1)$ is isomorphic to the additive group $\mathbb{Z}$.

Theorem 9.8 shows how to associate each homotopy class of loops in $S$ with an integer by the homotopy degree. The homotopy degree $d: \pi_1(S, 1) \to \mathbb{Z}$ is defined by $d(\gamma) = 1 \times 1 \in I \ni \gamma(0) \mapsto \gamma(s)$ for some $s$ in $I$.

\[ d(\gamma) = \begin{cases} 1 & \text{if } \gamma \text{ is not nullhomotopic} \\ 0 & \text{if } \gamma \text{ is nullhomotopic} \end{cases} \]

Thus the equivalence classes $[\gamma]$ and $[\delta \gamma]$ are equivalent if and only if $d(\gamma) = d(\delta \gamma)$. The nullhomotopy of $\gamma$ is equivalent to the nullhomotopy of $\delta \gamma$ if and only if $d(\gamma) = d(\delta \gamma)$. Therefore, the number of covering paths that pass through a given point $x$ is the same as the number of covering paths that pass through $y$. Since $F(x) = 0$, then $F(0) = 0$. For all $s$ in $I$, the same occurs for each value of $s$. Since $F(0) = 0$, then $F(s) = 0$ for all $s$ in $I$. Therefore, the same number of covering paths must be an integer. Since $F(x)$ is connected, the same number of covering paths must be an integer.
It remains to be proved that $d$ is a homomorphism. For $[\sigma], [\tau] \in \pi_1(S^1, 1)$, let $\tilde{\sigma}$ and $\tilde{\tau}$ denote the unique covering paths of $\sigma$ and $\tau$ beginning at 0. Then the path $g: I \to \mathbb{R}$ defined by

$$
g(t) = \begin{cases} 
\tilde{\sigma}(2t) & 0 \leq t \leq 1/2 \\
\tilde{\sigma}(1) + \tilde{\tau}(2t - 1) & 1/2 \leq t \leq 1
\end{cases}
$$

is the covering path of $\sigma \ast \tau$ with initial point 0. Thus

$$
deg(\sigma \ast \tau) = g(1) = \tilde{\sigma}(1) + \tilde{\tau}(1) = deg(\sigma) + deg(\tau).
$$

Hence

$$
d([\sigma] \ast [\tau]) = d([\sigma \ast \tau]) = deg(\sigma \ast \tau)
= deg(\sigma) + deg(\tau) = d([\sigma]) + d([\tau]).
$$

Thus $d$ is an isomorphism from $\pi_1(S^1, 1)$ onto $\mathbb{Z}$.

The covering projection $p: \mathbb{R} \to S^1$ has been instrumental in our computation of $\pi_1(S^1)$. The relevant properties of this map have been generalized to define an important class of such functions $p: E \to B$ from a covering space $E$ to a base space $B$ for which analogues of the Covering Path Property and Covering Homotopy Property can be established. The fundamental group is used to determine which spaces are covering spaces for a given space $B$. More complete information about covering spaces can be found in the Suggestions for Further Reading at the end of the chapter.

**EXERCISE 9.3**

1. Explain in detail why the loop $\mu_k: I \to S^1$ defined by

$$
\mu_k(t) = p(kt), \quad t \in I,
$$

has degree $k$, for each integer $k$.

2. Complete the inductive definition of the covering homotopy in the proof of the Covering Homotopy Property (Theorem 9.7).

3. Consider $S^1$ as the set $z = x + iy$ of complex numbers having modulus 1. Then the covering projection $p: \mathbb{R} \to S^1$ is, by definition of the exponential function for complex variables,

$$
p(t) = \cos 2\pi t + i \sin 2\pi t = e^{2\pi i t}, \quad t \in \mathbb{R}.
$$