Exam 3 Calculus 3 10/25/2002

Each problem is worth 10 points. Show adequate justification for full credit. Please circle all answers and keep your work as legible as possible. It takes two to tango.

1. State the definition of the partial derivative with respect to y of a function f(x,y).

$$f_y(x,y) = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

2. Show that $\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$ does not exist.

If we approach from $x=0$ direction we get,

$$\lim_{x \to 0} \frac{0^2}{0^2+y^2} = 0$$

If we approach from $x=y$ direction we get,

$$\lim_{x \to y} \frac{x^2}{x^2+x^2} = \lim_{x \to y} \frac{x^2}{2x^2} = \frac{1}{2}$$

So, limit has different values for different direction of approach and hence does not exist.

Great

3. Let $f(x,y) = 1 + 2x\sqrt{y}$. Find the directional derivative of f at the point (3,4) in the direction of the vector v = <4,-3>.

First we need to find the unit vector.

$\vec{u} = \frac{v}{|v|}$ $\quad |v| = \sqrt{4^2+(-3)^2} \rightarrow |v| = \sqrt{25} \rightarrow |v| = 5$

$\vec{u} = \langle \frac{4}{5}, -\frac{3}{5} \rangle$

Excellent

$f(x,y) = 1 + 2x(y^{\frac{1}{2}})$

$f_x(x,y) = 2\sqrt{y}$ $\qquad f_y = 2(\frac{1}{2})y^{-\frac{1}{2}}x \rightarrow f_y = \frac{x}{\sqrt{y}}$

$f_x(3,4) = 2\sqrt{4} \rightarrow f_x = 4$ $\qquad f_y(3,4) = \frac{3}{\sqrt{4}} \rightarrow f_y = \frac{3}{2}$

grad $f = \nabla f = \langle 4, \frac{3}{2} \rangle$

$D_u = \nabla f \cdot \vec{u}$

$D_u = \langle 4, \frac{3}{2} \rangle \cdot \langle \frac{4}{5}, -\frac{3}{5} \rangle \rightarrow D_u = 4(\frac{4}{5}) + \frac{3}{2}(-\frac{3}{5})$

$D_u = \frac{16}{5} - \frac{9}{10}$

$D_u = \frac{32}{10} - \frac{9}{10}$

$\boxed{D_u = \frac{23}{10}}$

4. Find an equation of the tangent plane to the surface $z = y^2 - x^2$ at the point (-4,5,9).

Equation of tangent plane: $z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$

$z = f(x,y) = y^2 - x^2$

$f_x(x,y) = -2x$ $\qquad f_y(x,y) = 2y$

$f_x(-4,5) = -2(-4)$ $\qquad f_y(-4,5) = 2(5)$

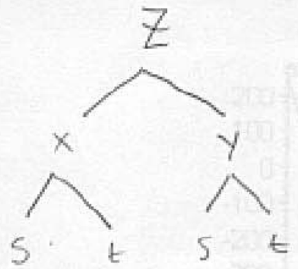
$f_x(-4,5) = 8$ $\qquad f_y(-4,5) = 10$

$\boxed{z - 9 = 8(x+4) + 10(y-5)}$

Wonderful

5. If z = f(x,y), x = x(s,t), and y = y(s,t), write the appropriate version of the chain rule for $\frac{\partial z}{\partial s}$.

Be careful to indicate clearly in your answer which derivatives are partials.



$$\frac{\partial z}{\partial s} = \left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial x}{\partial s}\right) + \left(\frac{\partial z}{\partial y}\right)\left(\frac{\partial y}{\partial s}\right)$$

10

6. Find the maximum rate of change of the function $f(x,y) = e^{y-x}$ at the point (2,-3) and the direction in which it occurs.

$$f(x,y) = e^{y-x}$$

$$f_x(x,y) = -e^{y-x} \qquad f_y(x,y) = e^{y-x}$$

$$\text{grad } \langle -e^{y-x}, e^{y-x} \rangle$$

10

$$\text{grad}(2,-3) = \langle -e^{-5}, e^{-5} \rangle$$

$$= |\text{grad}||u| \cos\theta$$

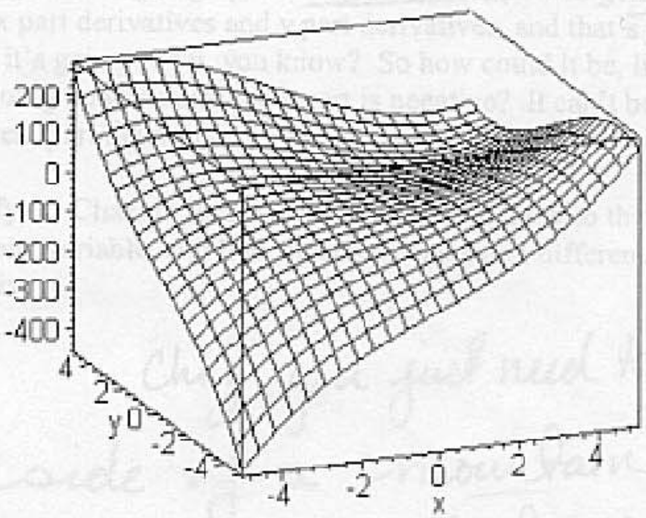
$$= |\text{grad}| \cos\theta,$$ in order to maximize, θ must be equal to 0, meaning the unit vector is in the same direction of the gradient

$$= \sqrt{(-e^{-5})^2 + (e^{-5})^2}$$

$$= \sqrt{2e^{-10}}$$

Excellent

7. The function $f(x,y) = x^3 + y^3 - 9xy + 27$ has critical points at (0,0) and (3,3). Classify these two critical points. The graph provided can serve as a guide, but it's up to you to demonstrate things.



$f_x(x,y) = 3x^2 - 9y$

$f_{xx}(x,y) = 6x$

$f_y(x,y) = 3y^2 - 9x$

$f_{yy}(x,y) = 6y$

$f_{xy}(x,y) = -9$

$$D = f_{xx} f_{yy} - (f_{xy})^2$$

$$D = (6x)(6y) - (-9)^2$$

D at $(0,0) = 0 - 81$ so $D < 0$ so the point (0,0) is a saddle point

D at $(3,3) = 36(3)(3) - 81$
$= 243$

So $D > 0$ so it could be a max or a min (3,3)

So we ✓ $f_{xx} = 6x$ at $(3,3) = 6(3) = 18 > 0$
and $f_{yy} = 6y$ at $(3,3) = 18 > 0$

Well done

So (3,3) is a max

8. Chaz is a calculus student at E.S.U., and he's having trouble with derivatives of functions of more than one variable. Chaz says "Man, this just makes no sense. I mean, I totally got it in calc 1, because slopes were just numbers and I can handle that, you know? So if the derivative was positive, it was going up, and negative meant it was going down, totally clear. But now there's, like, x part derivatives and y part derivatives, and that's just too strange. I mean, either it's going up or it's going down, you know? So how could it be, like, both going up if the x part is positive and going down 'cause the y part is negative? It can't be doing both, so how can there be different part derivatives?"

Clarify for Chaz, in terms he can understand, how to think about derivatives of functions of more than one variable, and how at a particular point different derivatives could be both positive and negative.

when dx   y is same

when dy   x is same

just because x goes up

y doesn't have to.   (mi north. temp)

When taking the partial deriviative, you are just interested in what is going on in the x direction, the y doesn't matter, so you treat the y as a constant. If the derivative in the x-direction is positive, the x-values will be increasing. Just because your x-values are increasing, that doesn't mean that your y-values have to increase as well. You find out what they do when you take the y-parital deriviative and ~~act~~ like the x-value is constant. It can be positive in one direction and negitive in another because you are working in three D. ~~x~~ and y are like if you are driving (and chart your distance & temp over time:) from Flordia to Maine in the winter. Just because your distance from Flordia (in miles) is increasing (like the x-direction), that doesn't mean that the temperature is increasing as well, in fact it is probably decreasing (like the y-direction).

Great

Good Example

Excellent

9. Show that the plane tangent to the surface $f(x,y) = x^2 - y^2$ at the point (x_0, y_0, z_0) has $-z_0$ as its z-intercept.

$f(x,y) = x^2 - y^2$

$f_x = 2x$ @ (x_0, y_0, z_0), $f_x = 2x_0$

$f_y = -2y$ $\quad f_y = -2y_0$

$z - z_0 = f_x(x - x_0) + f_y(y - y_0)$

$z - z_0 = 2x_0(x - x_0) - 2y_0(y - y_0)$

$z - z_0 = 2x_0 x - 2x_0^2 - 2y_0 y + 2y_0^2$

$= 2x_0 x - 2y_0 y - 2(x_0^2 - y_0^2)$

$\boxed{z = x^2 - y^2 \quad z_0 = x_0^2 - y_0^2}$ Yes!

and at z intercept, x and y are 0 so:

$z - z_0 = -2z_0$

$\therefore \boxed{z = -z_0}$

Excellent!

10. Consider the paraboloid $f(x,y) = x^2 + y^2$ and the collection of paraboloids $g(x,y) = -(x-1)^2 - (y-2)^2 + c$ for various values of the constant c. Exactly one of these g's should be tangent to f. For which value of the constant c will g and f be tangent?

Tangent means matching derivatives, so:

$f_x(x,y) = 2x \qquad g_x(x,y) = -2(x-1)$

$f_y(x,y) = 2y \qquad g_y(x,y) = -2(y-2)$

$2x = -2x+2 \Rightarrow 4x = 2 \Rightarrow x = \frac{1}{2}$

$2y = -2y+4 \Rightarrow 4y = 4 \Rightarrow y = 1$

So tangency occurs at $(\frac{1}{2}, 1)$, and $f(\frac{1}{2},1) = (\frac{1}{2})^2 + (1)^2 = \frac{5}{4}$, so we need g's height to match, or

$$g(\tfrac{1}{2},1) = -(\tfrac{1}{2}-1)^2 - (1-2)^2 + c$$

$$\frac{5}{4} = -\frac{1}{4} - 1 + c$$

$$c = \frac{5}{2}$$

Extra Credit (up to 5 points possible):
Prove the formula $\boxed{\text{grad}(f \cdot g)} = \boxed{f \cdot \text{grad } g + g \cdot \text{grad } f}$, where f and g are differentiable functions of the two variables x and y..

+5

$\boxed{\langle fg_x + gf_x, fg_y + gf_y \rangle}$ $=$

$$\boxed{\begin{aligned} &f \cdot \langle g_x, g_y \rangle + g \cdot \langle f_x, f_y \rangle \\ &\langle fg_x, fg_y \rangle + \langle gf_x, gf_y \rangle \\ &\langle fg_x + gf_x, fg_y + gf_y \rangle \end{aligned}}$$

last 2 lines are =

Yes!