Exam 5 Calculus 3 12/6/2002

Each problem is worth 10 points. Show adequate justification for full credit. Please circle all answers and keep your work as legible as possible. Don't forget the +C.

1. Give an example of a vector field whose divergence is 7.

$$\operatorname{div} \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 7$$

one set of numbers that can satisfy this is:

$$2 + 3 + 2 = 7$$

Nice!

now, anti-differentiate to find $\vec{F}$:

$$\vec{F} = \langle 2x + C,\ 3y + C,\ 2z + C \rangle$$

2. If F(x,y,z) = xy **i** + yz **j** + zx **k**, find ∇×F.

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \qquad \vec{F} = \langle xy, yz, zx \rangle$$

$\nabla \times \vec{F}$ is like curl:

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & zx \end{vmatrix} = \frac{\partial(zx)}{\partial y}\vec{i} + \frac{\partial(xy)}{\partial z}\vec{j} + \frac{\partial(yz)}{\partial x}\vec{k} - \left(\frac{\partial(yz)}{\partial z}\vec{i} + \frac{\partial(zx)}{\partial x}\vec{j} + \frac{\partial(xy)}{\partial y}\vec{k} \right)$$

$$= 0\vec{i} + 0\vec{j} + 0\vec{k} - y\vec{i} - z\vec{j} - x\vec{k}$$

Good!

$$= \langle -y, -z, -x \rangle$$

3. Find a potential function for the vector field xy i + yz j + zx k, or clearly explain (to Chaz, our faithful idiot from E.S.U.) how you know that no such potential function exists.

$$\overset{f_x}{xy\vec{i}} + \overset{f_y}{yz\vec{j}} + \overset{f_z}{zx\vec{k}}$$

To see if a pot. funct. exists, the curl of the function can be found

$\langle xy, yz, zx\rangle$

$$\text{curl}\,\vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\rangle \times \langle xy, yz, zx\rangle$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & zx \end{vmatrix} = -y\vec{i} - z\vec{j} - x\vec{k}$$

The curl does not equal zero, and so no potential function exists, for if a vector field has a potential function then the curl will be zero.

Great!

4. If F(x,y,z) = y sin x i - cos x j, find $\int_C F \cdot d\,r$ where C is the portion of the graph of y = cos x beginning at (0,1) and ending at (π, -1).

$$\vec{F}(x,y) = \langle y\sin x, -\cos x\rangle$$

$= -y\cos x$ is a potential function

$$-(-1)(\cos(\pi)) - (-(1)(\cos(0))$$

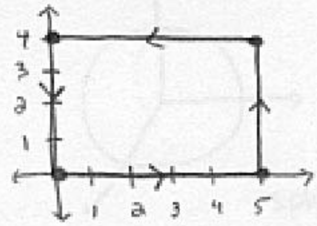
$$\cos(\pi) + \cos(0)$$

$$= 0$$

Well done

5. Compute $\int_C \mathbf{F} \cdot d\,\mathbf{r}$ for the vector field $\mathbf{F}(x,y)=(x^2y - 1)\mathbf{i} - (y^2)\mathbf{j}$ where C is the rectangular path beginning at (0,0), proceeding to (5,0), then (5,4), then (0,4), and finally back to (0,0).

$$F(x,y) = \langle \overset{P}{x^2y-1}, \overset{Q}{-y^2} \rangle$$



$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Closed Path so use Green's theorem.

$$= \iint_D (0 - x^2)\,dA$$

$$= \iint -x^2 \; dA$$

Great!

$$= \int_0^4 \int_0^5 -x^2 \; dx\,dy$$

$$= \int_0^4 \left[-\frac{x^3}{3} \right]_0^5 dy$$

$$= \int_0^4 \left[-\frac{125}{3} \right] dy$$

$$= \left[-\frac{125x}{3} \right]_0^4$$

$$= \boxed{-\frac{500}{3}}$$

6. If $\mathbf{F}(x,y,z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, find $\iint_S \mathbf{F} \cdot d\,\mathbf{S}$ where S is the surface of a sphere with radius 2 centered at the origin.

Divergence Theorem $\quad \iint_S \vec{F} \cdot d\vec{s} = \iiint_E \operatorname{div} \cdot \vec{F}$

$\vec{F} = \langle x, y, z \rangle$

$\operatorname{div} F = 1 + 1 + 1$
$= 3$

$= \iiint_E 3 \, dV$

$= 3 \iiint_E 1 \, dV$ a triple integral w/ 1 as integrand gives you volume, volume of a sphere $= \frac{4}{3}\pi r^3$

$= \cancel{3}\left(\frac{4}{\cancel{3}}\pi r^3\right)$

$= 4\pi 2^3$

$= 32\pi$

Nice Job!

7. Show that if f(x,y,z) is a function with continuous second order partial derivatives, then curl(grad f) = 0. Make clear how you use the continuity requirement.

$$\text{grad } f = \langle f_x, f_y, f_z \rangle$$

$$\text{curl}(\text{grad} f) = \nabla \times (\text{gradient } f)$$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix} = (f_{zy} - f_{yz})\vec{i} - (f_{zx} - f_{xz})\vec{j} + (f_{yx} - f_{xy})\vec{k}$$

if function has continuous second order partial derivatives then there second order partials should be equal

$$(f_{zy} = f_{yz},\ f_{xy} = f_{yx},\ f_{xz} = f_{zx})$$

THUS $\text{curl}(\text{grad} F) = 0\hat{i} - 0\hat{j} + 0\vec{k} = 0$

Beautiful!

8. Let $\mathbf{F}(x,y) = x^2\,\mathbf{i} + xy\,\mathbf{j}$. Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is an arc of a circle with radius 2 centered at the origin beginning at (2,0) and traversing n quadrants [if you have trouble with the n quadrants part, first try it for just 1 quadrant as a warm up].

$\vec{F} = \langle x^2, xy \rangle$

$P = x^2 \quad Q = xy$

Parametrize:
circle, rad 2

$\begin{cases} x = r\cos t \\ y = r\sin t \end{cases}$

$\begin{cases} x = 2\cos t \\ y = 2\sin t \end{cases}$

$0 \le t \le \frac{\pi}{2}n$

$r(t) = \langle 2\cos t, 2\sin t \rangle$

Build $\vec{F}(\vec{r}(t))$

$\vec{F}(x(t), y(t)) =$

$\vec{F} = \langle (2\cos t)^2, (2\cos t)(2\sin t) \rangle$

$\vec{F} = \langle 4\cos^2 t, 4\cos t\sin t \rangle$

Build $r'(t)$

$\langle -2\sin t, 2\cos t \rangle$

$$\int_C F \cdot dr = \int_0^{\frac{\pi}{2}n} \langle 4\cos^2 t, 4\cos t\sin t \rangle \cdot \langle -2\sin t, 2\cos t \rangle$$

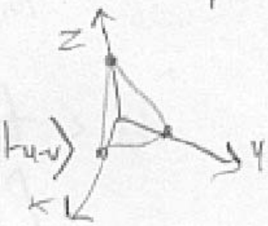
$$= \int_0^{\frac{\pi}{2}n} \left(-8\sin t\cos^2 t + -8\sin t\cos^2 t \right)$$

$$= \int_0^{\frac{\pi}{2}n} (0) = 0$$

Wonderful.

9. If $\mathbf{F}(x,y,z) = xze^y\,\mathbf{i} - xze^y\,\mathbf{j} + z\,\mathbf{k}$, find $\iint_S \mathbf{F}\cdot d\mathbf{S}$ where S is the part of the plane x+y+z=1 in the first octant with downward orientation. Feel free to stop at the point when you have an integral set up!

$z = 1-x-y$



$\vec{F} = \langle xze^y, -xze^y, z\rangle$

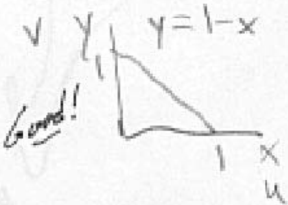
$x = u$

$y = v$

$z = 1-u-v$

$\vec{F}(\vec{r}) = \langle u(1-u-v)\cdot e^v, -u(1-u-v)e^v, 1-u-v\rangle$

$\vec{r} = \langle u, v, 1-u-v\rangle$



Good!

$\vec{r}_u = \langle 1, 0, -1\rangle$

$\vec{r}_v = \langle 0, 1, -1\rangle$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = (0\hat{i} + 0\hat{j} + \hat{k}) - (0\hat{k} - \hat{i} - \hat{j})$$

$= \langle 1,1,1\rangle \rightarrow$ down: $\langle -1,-1,-1\rangle$ yes!

$$\int_{u=0}^{1}\int_{v=0}^{1-u} \langle -1,-1,-1\rangle \cdot \langle \underbrace{(u-u^2-uv)e^v, (-u+u^2+uv)e^v}_{\text{cancel}}, 1-u-v\rangle \, dv\,du$$

$$\boxed{\int_{u=0}^{1}\int_{v=0}^{1-u} -(1-u-v)\, dv\,du}$$

Great Job!

10. On Quiz 4 the problem "Let $\mathbf{F}(x,y) = -y\mathbf{i} + x\mathbf{j}$. Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ for C the line segment beginning at (1,0) and ending at (2,3)." appeared, and the answer then was 3. How does it affect the result if we change the final point to (a,b)?

$(1,0)$ $\quad x = 1 + (a-1)t$

(a,b) $\quad y = bt$

$0 \le t \le 1$

$\vec{F}(\vec{r}(t)) = \langle -bt, 1+(a-1)t \rangle$

$r'(t) = \langle a-1, b \rangle$

$bt - bt + a + b + b(a-1)t$

$b(\cancel{t} - \cancel{at} + 1 + \cancel{at} - \cancel{t}) = b$

$\int_0^1 b\,dt$

$bt\Big|_0^1 = \boxed{b}$

Excellent!

y value of ending point is = to $\int F \cdot dr$!!

Very Exciting!!