Exam 1 Differential Equations 2/6/04

Each problem is worth 10 points. For full credit indicate clearly how you reached your answer.

1. Which of the following are differential equations? Circle all that are.

a) $\mathbf{X} = (\mathbf{A} - \lambda \mathbf{I})$

b) $y' = 3yt$

c) $\dfrac{dy}{dt} + f(y)g(t) - 3 \cdot \dfrac{y}{30+t}$ ← not a differential equation no equal sign

d) All men are mortal, and Socrates is a man, therefore all men are Socrates.

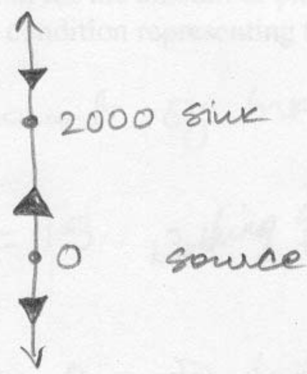
e) $y = Ae^{-3t} - B$

2. Sketch the phase line for the differential equation $\dfrac{dP}{dt} = 0.3P\left(1 - \dfrac{P}{2000}\right)$, and identify any equilibrium point(s) as sources, sinks, or nodes.

$0.3P\left(1 - \frac{P}{2000}\right) = 0$

$P = 0$ $\quad 1 - \frac{P}{2000} = 0$

$P = 2000$



Great

3. The differential equation for the temperature of a differential equations book placed in a 2000° furnace is $\frac{dH}{dt} = k(2000 - H)$, at least up until around where H(t) = 451°. This equation has general solutions of the form $H(t) = 2000 - Ae^{-kt}$. If a book has a temperature of 70° when first thrown into the furnace, and two minutes later has heated up to 200°, find a particular solution (with values accurate to two significant figures) fitting this situation.

$t=0 \quad T_{book} = 70 \quad \Rightarrow \quad H(0) = 2000 - Ae^{0} = 2000 - A = 70$

$\Rightarrow A = 1930$

$t=2 \quad T_{book} = 200 \quad \Rightarrow \quad H(2) = 2000 - 1930\, e^{-2k} = 2000$

$1800 = 1930\, e^{-2k}$

$0.07 = 2k \Rightarrow \underline{k = 0.035}$

$\underline{H(t) = 2000 - 1930 \cdot e^{-0.035t}}$

Great

4. Jon's water conditioner has been sabotaged so that it emits water with 5 grams of pink ink in every gallon of water. If Jon is running water at a rate of 3 gallons per minute into his 10 gallon sink, and the sink is half full of clean water before the ink solution begins to come out, write a differential equation for the amount of pink ink in the sink t minutes after the ink starts to come out, and an initial condition representing the situation.

let the amount of ink is S, then

$$\frac{dS}{dt} = \underbrace{5\,g/_{gal} \cdot 3}_{\text{ink in per min}} - \underbrace{3 \cdot \frac{S}{5\,gal}}_{}$$

Great

remain at 5 gal, then 3 gal in, gal out.

$$\Rightarrow \frac{dS}{dt} = 15 - \frac{3 \cdot S}{5},$$

with $\underline{t=0,\ S=0}$ as initial-value.

5. Find a general solution to the differential equation $\frac{dy}{dt} = \beta y - \alpha$. Separable!

$$\int \frac{1}{\beta y - \alpha}\,dy = \int dt$$

$$\frac{1}{\beta}\ln|\beta y - \alpha| = t + C$$

Different C

$$\ln|\beta y - \alpha| = \beta t + C$$

A's sign absorbs the absolute value.

$$\beta y - \alpha = Ae^{\beta t}$$

$$y = \frac{Ae^{\beta t} + \alpha}{\beta}$$

6. Show that if $f(x)$ is an integrable function, $g(x)$ is a differentiable function, and $u = g(x)$, then $\int f(g(x)) \cdot g'(x)\,dx = \int f(u)\,du$

Proof: Well, let $F(X)$ be a function for which $F'(X) = f(X)$.
By the chain rule, we get:

$$[F(g(X))]' = F'(g(X)) \cdot g'(X)$$

Antidifferentiating, we get:

$$F(g(X)) + c = \int F'(g(X)) \cdot g'(X)\,dX$$

And if we let $g(X) = u$, then we get

$$\begin{aligned}\int F'(g(X)) \cdot g'(X)\,dX &= F(g(X)) + c \\ &= F(u) + c \\ &= \int F'(u)\,du \\ &= \int f(u)\,du. \square\end{aligned}$$

Nice!

7. Find a general solution to the differential equation $\frac{dy}{dt} = e^{2t} + 5y$.

This is linear

$\frac{dy}{dt} - 5 \cdot y = e^{2t}$

$\mu = e^{\int -5\,dt}$

$\mu = e^{-5t}$

$\frac{dy}{dt} \cdot e^{-5t} - 5y \cdot e^{-5t} = e^{2t} \cdot e^{-5t}$) add e^{-3t}

$\int (y \cdot e^{-5t})' = \int e^{-3t}\,dt$

$u = -3t$

$du = -3\,dt$

$\frac{du}{-3} = dt$

$y \cdot e^{-5t} = -\frac{1}{3} e^{-3t} + C$

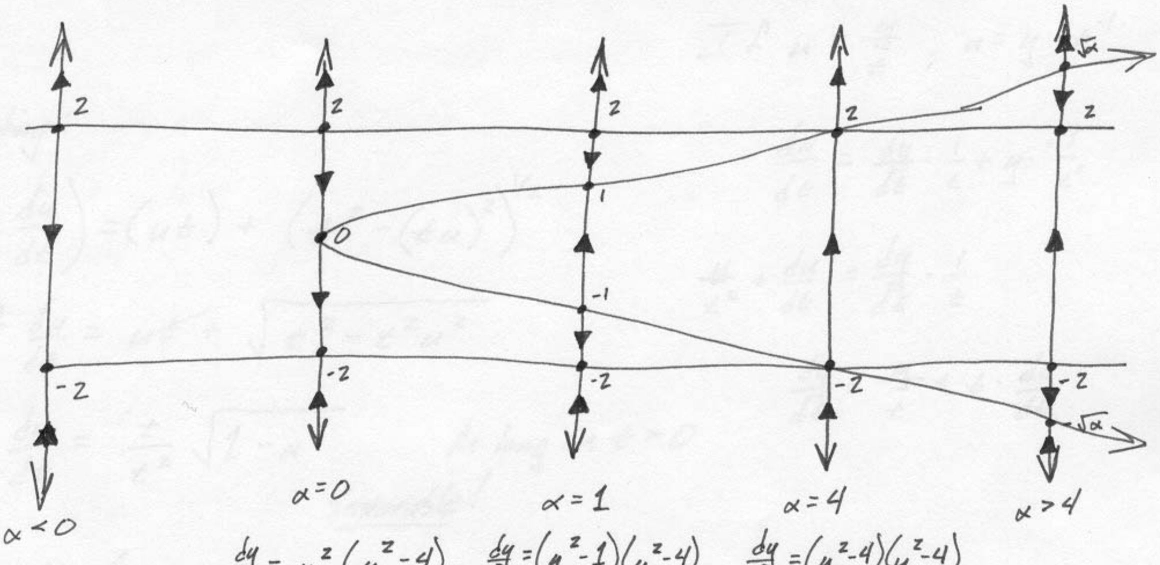
$y = \dfrac{-\frac{1}{3} e^{-3t} + C}{e^{-5t}}$

Nice Job

$y = -\frac{1}{3} e^{-3t} (e^{+5t}) + C \cdot e^{+5t}$

$\boxed{y(t) = -\frac{1}{3} e^{2t} + C \cdot e^{5t}}$

8. **Sketch the bifurcation diagram** for the differential equation $\frac{dy}{dt} = (y^2 - \alpha)(y^2 - 4)$. Include direction arrows on the phase lines and make clear the exact α values where bifurcations occur.



$\frac{dy}{dt} = y^2(y^2-4)$ $\quad$ $\frac{dy}{dt} = (y^2-1)(y^2-4)$ $\quad$ $\frac{dy}{dt} = (y^2-4)(y^2-4)$

Bifurcations when $\alpha = 0$, from two to three then four eq.,
and $\alpha = 4$, temporarily just two eq.
and the arrows rearrange a bit.

9. Solve the initial value problem $t\dfrac{dy}{dt} = y + \left(t^2 - y^2\right)^{1/2}$, $y(1) = 0$ by using the substitution

$u = \dfrac{y}{t}$.

If $u = \dfrac{y}{t}$, $u = y \cdot t^{-1}$

$$\frac{du}{dt} = \frac{dy}{dt}\cdot\frac{1}{t} + y\cdot\frac{-1}{t^2}$$

$$\frac{y}{t^2} + \frac{du}{dt} = \frac{dy}{dt}\cdot\frac{1}{t}$$

$$\frac{dy}{dt} = \frac{y}{t} + t\cdot\frac{du}{dt}$$

So substituting:

$$t\left(\frac{y}{t} + t\frac{du}{dt}\right) = (ut) + \left(t^2 - (tu)^2\right)^{1/2}$$

$$\cancel{y} + t^2\frac{du}{dt} = \cancel{ut} + \sqrt{t^2 - t^2u^2}$$

$$\frac{du}{dt} = \frac{t}{t^2}\sqrt{1-u^2}$$ As long as $t > 0$

Separable!

$$\int \frac{1}{\sqrt{1-u^2}}\,du = \int \frac{1}{t}\,dt$$

$$\arcsin u = \ln t + C$$

$$u = \sin(\ln t + C)$$

$$\frac{y}{t} = \sin(\ln t + C)$$

$$y = t\cdot\sin(\ln t + C)$$ General Solution.

$$(0) = (1)\sin(\ln(1) + C)$$

$0 = \sin C$ So $C = 0$ works, as do multiples of π.

$$y = t\sin(\ln t)$$ Particular Solution.

10. Somebody tells you that if two functions are each solutions to a differential equation, then their product must also be a solution to that differential equation. Are they right? Why or why not?

Nope. For a counterexample, notice that $y' = y$ has $y_1 = e^t$ as a solution and $y_2 = 2e^t$ as another solution, but $y_3 = y_1 \cdot y_2 = e^t \cdot 2e^t = 2e^{2t}$ is <u>not</u> a solution.