Exam 2 Take-home Portion **Differential Equations** **3/17/06**

Each problem is worth 10 points. You may freely consult our textbook or any notes you generated prior to receiving this exam, but may **not** consult with any living being directly or indirectly, nor outside resources of any sort (except of course spreadsheets or software not involving internet access).

6. Use Laplace transforms to find a solution to the differential equation $\frac{dy}{dt} = e^{3t} + 5y$ subject to the initial condition $y(0) = D$.

First apply the transform:

$$\mathcal{L}\left(\frac{dy}{dt}\right) = \mathcal{L}(e^{3t}) + \mathcal{L}(5y)$$

$$s \cdot \mathcal{L}(y) - y(0) = \frac{1}{s-3} + 5\mathcal{L}(y)$$

$$\mathcal{L}(y)(s-5) = \frac{1}{s-3} + D$$

$$\mathcal{L}(y) = \frac{1}{(s-3)(s-5)} + \frac{D}{s-5}$$

$$\mathcal{L}(y) = \frac{-1/2}{s-3} + \frac{1/2}{s-5} + \frac{D}{s-5}$$

Now the inverse transform:

$$y = -\frac{1}{2}e^{3t} + \frac{1}{2}e^{5t} + De^{5t}$$

Partial Fractions:

$$\frac{1}{(s-3)(s-5)} = \frac{A}{s-3} + \frac{B}{s-5}$$

$$1 = A(s-5) + B(s-3)$$

If $s=5$: $1 = 2B$, $B = 1/2$

If $s=3$: $1 = -2A$, $A = -1/2$

7. Consider the system of differential equations for two populations:

$$\frac{dx}{dt} = 8x - 2x^2 - 4xy$$

$$\frac{dy}{dt} = 9y - 5xy - 3y^2$$

a) Use Euler's method with $\Delta t = 0.05$ to approximate $x(3)$ and $y(3)$ if $x(0) = 0.7$ and $y(0) = 2.6$.

b) Use Euler's method with $\Delta t = 0.05$ to approximate $x(3)$ and $y(3)$ if $x(0) = 0.5$ and $y(0) = 2.6$.

c) Use Euler's method with $\Delta t = 0.05$ to approximate $x(3)$ and $y(3)$ if $x(0) = 0.86$ and $y(0) = 1.57$.

d) Comment on the equilibria of this system. Do they appear to be sources, sinks, or nodes?

a) $x(3) \approx 6.7246 \times 10^{-6}$
$y(3) \approx 2.9997$

b) $x(3) \approx 2.6863 \times 10^{-6}$
$y(3) \approx 2.9999$

c) $x(3) \approx 2.6092$
$y(3) \approx 0.3105$

d) There are equilibria at $(0,0)$, $(0,3)$, $(4,0)$ and $\left(\frac{6}{7}, \frac{11}{7}\right)$

$(0,0)$ is clearly a source

$(0,3)$ and $(4,0)$ are clearly sinks

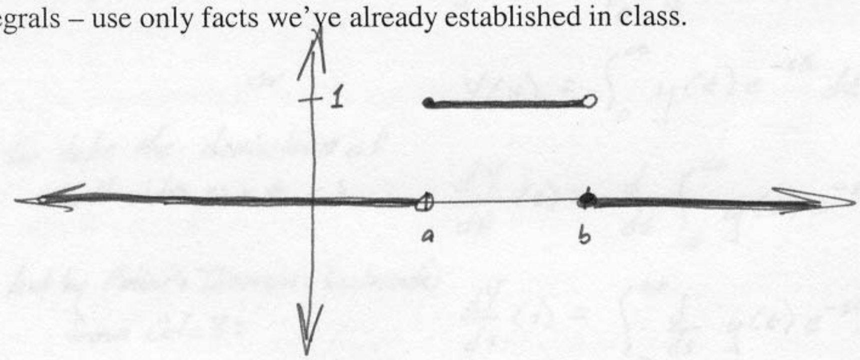
$(6/7, 11/7)$ is a bit stranger than any equilibria we've dealt with previously. It's like a sink from below left or above right, but like a source to below right or above left. Calling it a node isn't unreasonable at this stage.

8. Do problem #14 from §2.4 in the text.

From an analytic standpoint, since we know $\hat{Y}_1$ is a solution, $\frac{d\hat{Y}_1}{dt}(t) = \hat{F}(\hat{Y}_1(t))$ holds $\forall\, t \in \mathbb{R}$. Then in particular it holds for $t_1 = t + t_0$, so $\frac{d\hat{Y}_1}{dt}(t_1) = \hat{F}(\hat{Y}_1(t_1))$, or $\frac{d\hat{Y}_1}{dt}(t+t_0) = \hat{F}(Y_1(t+t_0))$.
But that just means $\frac{dY_2}{dt}(t) = \hat{F}(Y_2(t))$ holds $\forall\, t \in \mathbb{R}$.

Or from a more qualitative standpoint, the Existence and Uniqueness Theorem on page 201 assures us that since the two solution curves intersect, then "their images are the same curves in the phase plane, and they differ only in their parametrizations."

9. Find the Laplace transform of the function $blip_{a,b}(t) = \begin{cases} 0, & \text{if } t \in (-\infty, a) \\ 1, & \text{if } t \in [a,b) \\ 0, & \text{if } t \in [b,\infty) \end{cases}$ **without** computing any integrals – use only facts we've already established in class.



First notice that $blip_{a,b}(t) = u_a(t) - u_b(t)$ for $0 > a > b$.

But from class we know $\mathcal{L}(u_a(t)) = \frac{e^{-as}}{s}$ and

$\mathcal{L}(u_b(t)) = \frac{e^{-bs}}{s}$ for $s > 0$

So then since the Laplace transform is linear,

$$\begin{aligned} \mathcal{L}(blip_{a,b}(t)) &= \mathcal{L}(u_a(t) - u_b(t)) \\ &= \mathcal{L}(u_a(t)) - \mathcal{L}(u_b(t)) \\ &= \frac{e^{-as}}{s} - \frac{e^{-bs}}{s} \end{aligned}$$

10. If you know that $y(t)$ has Laplace transform $Y(s)$, with some domain $S \subseteq \mathbb{R}$, what can you say about the Laplace transform of $t \cdot y(t)$?

Well, we know $\mathcal{L}(y(t)) = \int_0^\infty y(t) e^{-st} dt$

or $Y(s) = \int_0^\infty y(t) e^{-st} dt$

So take the derivative of both sides w.r.t. s:

$$\frac{dY}{ds}(s) = \frac{d}{ds} \int_0^\infty y(t) e^{-st} dt$$

And by Fubini's Theorem (backwards) from Calc 3:

$$\frac{dY}{ds}(s) = \int_0^\infty \frac{d}{ds} y(t) e^{-st} dt$$

$$\frac{dY}{ds}(s) = \int_0^\infty y(t) \cdot -t e^{-st} dt$$

$$\frac{dY}{ds}(s) = -\int_0^\infty t\, y(t)\, e^{-st} dt$$

$$\boxed{-\frac{dY}{ds}(s) = \mathcal{L}(t \cdot y(t))}$$