Examlet 3 **Foundations of Advanced Math** **3/21/08**

1. a) State the definition of an increasing function f: $\mathbb{R} \to \mathbb{R}$.

A function $f: \mathbb{R} \to \mathbb{R}$ is increasing iff whenever $x > y$, $f(x) > f(y)$.

Good!

b) State the definition of an odd function f: $\mathbb{R} \to \mathbb{R}$.

A function $f: \mathbb{R} \to \mathbb{R}$ is odd iff $f(-x) = -f(x)$ for all $x \in \mathbb{R}$.

Great!

2. Let $f: A \to B$ be invertible. Show that $f^{-1} \circ f = I_A$.

by def. of an inverted function ⇒

$f^{-1}: B \to A$

$f(a) = b \Leftrightarrow f^{-1}(b) = a$

$I_A(a) = a$ for $I_A: A \to A$

$f^{-1}(f(a)) = f^{-1}(b) = a$

for two functions to be equal, thier domains + codomains must be the same and

$f(x) = g(x) \ \forall x \in A$. (if A is the domain)

$f^{-1} \circ f : A \to A$ and $I_A : A \to A$

and $f^{-1}(f(a)) = a \ \forall a \in A$

$I_A(a) = a \ \forall a \in A$

Wonderful!

Therefore, $f^{-1} \circ f = I_A$ by definition of equality of functions.

3. Let $f: A \to B$ and $g: B \to C$ be injective functions. Show that $g \circ f$ is injective.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "h"'] & B \arrow[r, "g"] & C \end{tikzcd}$$

Let's say $h = g \circ f$

Taking two arbitrary elements of A, a_1 and a_2, let

$$h(a_1) = h(a_2)$$

$g \circ f(a_1) = g \circ f(a_2)$ By def. of h

$g(f(a_1)) = g(f(a_2))$ By def. of compositions

$f(a_1) = f(a_2)$ B/c g is injective

$a_1 = a_2$ B/c f is injective

Thus, since for function h $(\forall a, b \in A)[h(a) = h(b) \Rightarrow a = b]$ h is inductive, as this is the definition of inductive.

Very Nice!

4. a) Show that $\mathbb{Z}$ is denumerable.

a set A is denumerable iff a bijection from $\mathbb{N} \to A$ exists. So take $f: \mathbb{N} \to \mathbb{Z}$

$$f(n) = \begin{cases} 0 & \text{if } n = 1 \\ n/2 & \text{if } n/2 \in \mathbb{N} \\ -\left(\frac{n-1}{2}\right) & \text{if } \frac{n-1}{2} \in \mathbb{N} \end{cases}$$

$1 \to 0$
$2 \to 1$
$3 \to -1$
$4 \to 2$
$5 \to -2$

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Excellent

f(n) is injective because the components 0, n/2, and $-\left(\frac{n-1}{2}\right)$ are injective as they are non-zero linear functions and there is no overlap among the components. f(n) is surjective as well since $\forall x \in \mathbb{Z}$ $\exists n \in \mathbb{N}$ $f(n) = x$. Thus, $\mathbb{Z}$ is denumerable.

there for f(n) is a bijection

b) Show that if A is uncountable and x is an object in A, then $A - \{x\}$ is uncountable.

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a set is uncountable if it is infinite and non-denumerable

S'pose that $A - \{x\}$ was in fact countable. since A is uncountable, we know that A is infinite, so $A - \{x\}$ must also be infinite. Then $A - \{x\}$ must be denumerable since it is countable. Say $f: \mathbb{N} \to A - \{x\}$ exists

slightly confused

$$\begin{cases} f(1) = \text{the first element in } A - \{x\} \\ \text{and } f(n+1) = \text{the smallest element in } A - \{x\} \text{ larger than } f(n). \end{cases}$$

If this is the case, we could then say let $g(n): \mathbb{N} \to A$ where $g(1) = \{x\}$ and $g(n) = f(n-1)$. But then we have a bijection from $\mathbb{N} \to A$, and A is uncountable so we have a contradiction. *Great reasoning!*

There fore our origional assumption that $A - \{x\}$ was countable was incorrect. Thus, $A - \{x\}$ is uncountable.

5. Let $\{A_i \mid i \in \mathbb{N}\}$ be an indexed family of sets, and suppose that A_i is bounded for every $i \in \mathbb{N}$. Let $Z_n = \{m \in \mathbb{N} \mid m \le n\}$. Show that $\bigcup_{i \in Z_n} A_i$ is bounded $\underbrace{\text{for all } n \in \mathbb{N}}_{\text{clue!}}$. Induction!

Well, let's induct on n, the number of things in our index set.

If $n=1$, $Z_1 = \{1\}$, so our union only includes things from A_1, which was bounded, and so that same bound works for the union.

Or just say "we did this on a Problem set."

If $n=2$, $Z_2 = \{1,2\}$, so $\bigcup_{i \in Z_2} A_i = A_1 \cup A_2$. Since A_1 was bounded, say by M_1, and A_2 was bounded, say by M_2, we have $\forall a \in A_1$, $|a| < M_1$ and $\forall a \in A_2$, $|a| < M_2$. But then if we let M be whichever of M_1 and M_2 is larger, we see $A_1 \cup A_2$ is bounded, since if $a \in A_1 \cup A_2$, we have $a \in A_1 \Rightarrow |a| < M_1 \le M$ or $a \in A_2 \Rightarrow |a| < M_2 \le M$.

Now s'pose it's true for $n=k$, so $\bigcup_{i \in Z_k} A_i$ is bounded. Then for $n=k+1$, we have $\bigcup_{i \in Z_{k+1}} A_i = \left(\bigcup_{i \in Z_k} A_i\right) \cup A_{k+1}$, with both of those sets bounded, so their union is bounded by the $n=2$ argument.

Thus by induction, the statement is true for all $n \in \mathbb{N}$. $\square$