Topology – Midterm Exam – Spring 2012

1. Let (X, $\mathscr{T}$) be a topological space. Let $A, B \subseteq X$.
   a. Prove that $(A \cap B)' \subseteq A' \cap B'$.
   b. Give an example of a topological space and sets A, B such that $A' \cap B' \subseteq (A \cap B)'$ is false.

2. Let $A = [0,1) \cup (2,3]$ be a subset of ($\mathbb{R}$, $\mathscr{H}$). Find each of the following:
   a. Cl (A)
   b. Int (A)
   c. Bd (A)
   d. Ext (A)

3. Let $\mathscr{B}$ be a base for a topological space (X, $\mathscr{T}$) and let $A \subseteq X$. Show that the collection $\{B \cap A : B \in \mathscr{B}\}$ is a base for some topology on A.

4. Let $f : X \to Y$ be a function and let $U \subseteq X$ and $V \subseteq Y$. Prove:
   a. $f(f^{-1}(V)) \subseteq V$
   b. $U \subseteq f^{-1}(f(U))$

5. Let $f : X \to Y$ be a function and let A and B be subsets of X. Prove that if f is one to one, then $f(A \cap B) = f(A) \cap f(B)$.

6. Let A be a subset of the space (X, $\mathscr{T}$). The set A is $\mathscr{T}$- open iff $\mathscr{T}_A \subseteq \mathscr{T}$.

7. Let (X, $\mathscr{T}$) be a topological space and let $A \subseteq X$. The set A is said to be *dense* in X provided that Cl (A) $= X$.

   Let D be dense in X and let U be an open subset of X.
   a) Prove that $U \cap D \neq \varnothing$
   b) Prove that if U is dense in X, then $U \cap D$ is dense in X.

8. Establish each of the following as true or false. If true, explain briefly why it is true. If false, give a counterexample.

a. If $f: X \to Y$ is an **onto** function and V is a non-empty subset of Y, then $f^{-1}(V)$ is a non-empty subset of X.

b. Any one-to-one, onto function between two discrete topological spaces is a homeomorphism.

c. If A is a subset of a topological space, then $A \subseteq A'$.

d. If A is a closed set in a topological space, then Bd $A \subseteq A$.

e. There exists a topological space (X, $\mathscr{T}$) such that there is no base for $\mathscr{T}$.

f. Every constant function is continuous regardless of the topologies on the domain and codomain.

g. If X and Y are homeomorphic topological spaces, then any one-to-one function from X onto Y is a homeomorphism.

h. If $f: X \to Y$ is a continuous function, then $f^{-1}(Y) = X$.

i. If a function $f: \mathbb{R} \to \mathbb{R}$ is $\mathscr{U} - \mathscr{H}$ continuous function then f is $\mathscr{U} - \mathscr{U}$ continuous.

j. If $f: X \to Y$ is an open function and V is a open subset of Y, then $f^{-1}(V)$ is an open subset of X.

9. Let $f:(X,\mathscr{T}) \to (Y,\mathscr{S})$. f is said to be a *closed* function if for each $\mathscr{T}-closed$ subset U of X, $f(U)$ is an $\mathscr{S}-closed$ subset of Y. Prove that $f:(X,\mathscr{T}) \to (Y,\mathscr{S})$ is a homeomorphism if and only if f is one-to-one, onto, continuous and *closed.*

10. Let $f:(X,\mathscr{T}) \to (Y,\mathscr{S})$. Prove: f is open if and only if $f[Int(A)] \subseteq Int(f[A])$ for all $A \subseteq \mathbf{X}$.